

Characterizations, extensions, and factorizations of Hilbert space operators

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PhD Thesis

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Introduction

Our aim in the present dissertation is to discuss problems of the operator theory of Hilbert spaces. More precisely, we investigate characterization, factorization, and extension problems of various classes of operators and suboperators, respectively. The dissertation is based on author's papers [52, 53, 60, 61, 62, 63]. The results of Chapter 1 and 2 are taken from the manuscripts [54, 55] of the author.

A central question of theory of unbounded operators is to characterize (essentially) selfadjoint operators among symmetric ones. For densely defined operators, the problem is addressed to by J. von Neumann [36]. Chapters 1 and 2 contain some revisions of Neumann's results. While Neumann's arguments (among others the Cayley transform) could be used only for densely defined symmetric operators acting on complex Hilbert spaces, our approach makes it possible to investigate the characterization problem also among not necessarily densely defined ones. At the same time, the real and complex Hilbert spaces can be treated jointly.

A celebrated theorem of J. von Neumann reads as follows: if T is any densely defined and closed operator between Hilbert spaces then T^*T is a positive selfadjoint operator. Nevertheless, easy example illustrate that closedness of T is sufficient but not necessary condition for the conclusion. In Chapter 4 we prove that T^*T always has a positive selfadjoint extension for a densely defined T . Further we give metric characterization of the selfadjointness of T^*T . Finally, we give an extension of a classical result of Friedrichs [18]: if A is any densely defined positive operator in a Hilbert space \mathfrak{H} , then there is a densely defined closable operator T , acting also in \mathfrak{H} , such that the factorizations $A = T^*T$ and $A_F = T^*T^{**}$ hold, where A_F denotes the well known Friedrichs extension of A .

In Chapter 5 we investigate suboperators which have closed range extensions. In particular, we characterize those positive operators which have positive selfadjoint extensions with closed range. Extremal extensions of suboperators are also described. The Moore–Penrose pseudoinverse of the Krein–von Neumann extension is constructed in case when it exists at all. Our main tool in this chapter (and also in the subsequent chapters) is the factorization method of Z. Sebestyén and J. Stochel [49] for giving the Krein–von Neumann extension of a not necessarily densely defined positive operator. We, therefore, briefly recall here the construction of this smallest possible positive selfadjoint extension.

In Chapter 6, continuing the investigations of [16], we give domain, kernel and range characterizations of the form sum of two positive operators. In addition, we establish

a criterion for the closedness of the range of the form sum and give the Moore–Penrose pseudoinverse in this case.

In Chapter 7 we give an extension of a classical result due to Krein [34] on biorthogonal expansions of compact operators which are symmetrizable with respect to a nondegenerate positive operator. Our approach makes essential use of the spectral expansion of an appropriate compact selfadjoint operator, the existence of which is due to Dieudonné [12].

Finally, in Chapter 8 we investigate Lebesgue-type decompositions of positive operators due to Ando [2]. We give a new construction for the Lebesgue decomposition of positive operators on Hilbert spaces with respect to each other. Our approach is similar to that of Kosaki: we use unbounded operator techniques and factorizations via two auxiliary Hilbert spaces associated to the positive operators in question.

I am deeply indebted to my supervisor Prof. Zoltán Sebestyén who made many valuable suggestions.

CHAPTER 1

Characterization of selfadjoint operators

Revision of von Neumann's characterization of selfadjoint operators among symmetric operators and its applications is our main purpose. Algebraic arguments we use goes back to Arens [4], contrary to the geometric nature of Cayley transform used by von Neumann [36]. We do not assume that the underlying Hilbert space is complex, nor that the corresponding symmetric operator is densely defined: it is a consequence.

The Hilbert spaces $\mathfrak{H}, \mathfrak{K}$ are real or complex (or quaternionic) and the operators A, B acting between them are not necessarily densely defined. In contrast they will be automatically of that size in case of selfadjointness.

In particular, a linear operator $A : \mathfrak{H} \rightarrow \mathfrak{H}$ with domain (and range) $\text{dom } A$ (resp. $\text{ran } A$) is called symmetric, in contrast to the usual assumption when densely definedness is at the same time assumed, when A satisfies the following identity

$$(Ax, y) = (x, Ay), \quad (x, y \in \text{dom } A).$$

Note that in the complex Hilbert space case iA is symmetric if A is skew-symmetric:

$$(Ax, y) + (x, Ay) = 0, \quad (x, y \in \text{dom } A).$$

It happens even in the fundamental case, the so called Stone Theorem [59], when the derivative A at zero of a strongly continuous one parameter unitary group is automatically skew-adjoint on a (real or complex) Hilbert space (see Corollary 1.12 below), i.e. $A^* = -A$ satisfies.

For a symmetric operator A on a real (or complex) Hilbert space we have the following statements

Lemma 1.1. *Let A be symmetric, not necessarily densely defined operator on a real or complex Hilbert space. Then for the kernel and the range of the corresponding operator satisfies*

$$\ker A \subseteq \{\text{ran } A\}^\perp.$$

Furthermore, the identity $\ker A = \{\text{ran } A\}^\perp$ implies

$$\{\text{dom } A\}^\perp \cap \text{ran } A = \{0\}.$$

Proof. Let y belong to $\ker A$, then $Ay = 0$ and

$$0 = (x, Ay) = (Ax, y) \quad (x \in \text{dom } A)$$

holds, therefore y belongs to $\{\text{ran } A\}^\perp$, indeed.

Otherwise, for $Ay \in \{\operatorname{dom} A\}^\perp$, when $y \in \operatorname{dom} A$, we have that

$$0 = (x, Ay) = (Ax, y) \quad (x \in \operatorname{dom} A)$$

once more so that $y \in \{\operatorname{ran} A\}^\perp = \ker A$, thus $Ay = 0$. \square

As a consequence a revised statement of M. H. Stone [58, Theorem 2.19] follows:

Proposition 1.2. *Let A be a symmetric operator on a real or complex Hilbert space with full range, i.e. $\operatorname{ran} A = \mathfrak{H}$. Then A is automatically selfadjoint with trivial kernel and bounded (selfadjoint) inverse.*

Proof. Direct consequence of Lemma 6.1 implies that $\ker A = \{0\}$ and that the inverse A^{-1} of the operator A is everywhere defined symmetric operator. The classical Hellinger-Toeplitz Theorem can be applied: A^{-1} is a bounded selfadjoint operator. Here, its inverse A is also selfadjoint, although not necessarily bounded (densely defined) operator. \square

In the language of the spectrum $\operatorname{Sp}(A)$ of the operator A we formulate the following

Theorem 1.3. *The symmetric operator A acting in a real or complex Hilbert space fulfills the following equivalent conditions for a real number $t \in \mathbb{R}$:*

- (i) A is selfadjoint and $t \notin \operatorname{Sp}(A)$.
- (ii) $t \notin \operatorname{Sp}(A)$.
- (iii) $\operatorname{ran}(A - tI) = \mathfrak{H}$.

Proof. It is obvious that (i) \Rightarrow (ii) \Rightarrow (iii). Assuming (iii), $A - tI$ is a full range symmetric operator so that Proposition 6.2 applies: it is a selfadjoint operator with bounded inverse formulated by (i). \square

1.1. Characterizations of selfadjoint operators

Assume for a moment that the linear operator A between the Hilbert spaces \mathfrak{H} and \mathfrak{K} is densely defined, then its adjoint A^* acts between \mathfrak{K} and \mathfrak{H} as follows:

$$\begin{aligned} \operatorname{dom} A^* &= \{z \in \mathfrak{K} : \sup\{|(Ax, z)|^2 : x \in \operatorname{dom} A, (x, x) \leq 1\} < \infty\}, \\ (Ax, z) &= (x, A^*z) \quad (x \in \operatorname{dom} A, z \in \operatorname{dom} A^*). \end{aligned}$$

Moreover, we have at the same time that

$$\begin{aligned} \ker A^* &= \{\operatorname{ran} A\}^\perp, \\ \operatorname{ran} A^* &= \{y \in \mathfrak{H} : \sup\{|(x, y)|^2 : x \in \operatorname{dom} A, (Ax, Ax) \leq 1\} < \infty\}. \end{aligned}$$

The range characterization here is taken from Sebestyén [44].

In the $\mathfrak{H} = \mathfrak{K}$ case, i.e. when A is an operator from \mathfrak{H} into \mathfrak{H} , A is selfadjoint when $A = A^*$ and skew-adjoint when $A^* = -A$.

The following statement goes back, in the linear relation case, to Arens [4, (2.02)].

Proposition 1.4. *Let $A : \mathfrak{H} \rightarrow \mathfrak{H}$ be a densely defined linear operator. The following statements are equivalent:*

- (i) A is selfadjoint.
- (ii) a) A is symmetric,
b) $\ker A = \{\operatorname{ran} A\}^\perp$,
c) $\operatorname{ran} A = \operatorname{ran} A^*$.

Proof. It is clear that (i) implies (ii). Conversely, thanks to (ii) a) we have that A^* extends A so that we should only check that $\operatorname{dom} A^* \subseteq \operatorname{dom} A$. Chosen z from $\operatorname{dom} A^*$ we find $y \in \operatorname{dom} A$ by (ii) c) such that $A^*z = Ay$. But $Ay = A^*y$ satisfies so that

$$(z - y) \in \ker A^* = \{\operatorname{ran} A\}^\perp = \ker A,$$

according to (ii) b). Therefore $z = (z - y) + y \in \operatorname{dom} A$, indeed. \square

Our next result gives a new characterization of selfadjointness of a symmetric operator which is not assumed apriori to be densely defined:

Theorem 1.5. *Let $A : \mathfrak{H} \rightarrow \mathfrak{H}$ be a linear operator. The following statements are equivalent:*

- (i) A is selfadjoint.
- (ii) a) A is symmetric,
b) $\ker A = \{\operatorname{ran} A\}^\perp$,
c) $\operatorname{ran} A = \{y \in \mathfrak{H} : \sup\{|(x, y)|^2 : x \in \operatorname{dom} A, (Ax, Ax) \leq 1\} < \infty\}$.

Proof. It is clear that (i) implies (ii). By assuming (ii), we check first that A is densely defined: for if z belongs to $\{\operatorname{dom} A\}^\perp$ then z also belongs to the range of A by assumption (ii) c). But if $z = Ay$ for some $y \in \operatorname{dom} A$, then we have

$$0 = (x, z) = (x, Ay) = (Ax, y), \quad (x \in \operatorname{dom} A),$$

by using that A is symmetric (by (ii) a)). It follows that y belongs to $\{\operatorname{ran} A\}^\perp = \ker A$ by (ii) b), hence that $z = Ay = 0$, indeed. Proposition 1.4 now applies by the range characterization (ii) c) of the adjoint operator A^* above. \square

1.2. Characterization of positive selfadjoint operators

Here we apply the former results to positive symmetric operators on real (or complex) Hilbert spaces. An operator $A : \mathfrak{H} \rightarrow \mathfrak{H}$ is called positive if $(Ax, x) \geq 0$ holds for each $x \in \operatorname{dom} A$.

Proposition 1.6. *Let $A : \mathfrak{H} \rightarrow \mathfrak{H}$ be not necessarily densely defined positive symmetric operator in a real or complex Hilbert space. The following statements are equivalent:*

- (i) A is selfadjoint.
- (ii) $\operatorname{Sp}(A)$ does not contain a negative real number.

(iii) $A + tI$ has full range for some positive real number t .

Proof. It is an easy exercise that (i) follows (ii): since A is selfadjoint, it is closed in particular as well as the operator $A + tI$. The latter one is also bounded from below by t whenever t is positive number. Therefore, the range of $A + tI$ is closed and dense at the same time according to the identities

$$\{0\} = \ker(A + tI) = \{\operatorname{ran}(A + tI)\}^\perp.$$

That means that $A + tI$ is of full range and thus $-t \notin \operatorname{Sp}(A)$ in view of Theorem 6.3. According again to Theorem 6.3, implications (ii) \Rightarrow (iii) and (iii) \Rightarrow (i) are clear. \square

Selfadjointness of T^*T was proved by von Neumann in case of closed densely defined operator T between complex Hilbert spaces [37]. Without the closedness assumption we have the following statement.

Theorem 1.7. *Let $T : \mathfrak{H} \rightarrow \mathfrak{K}$ be a densely defined operator between real or complex Hilbert spaces. The following statements are equivalent:*

- (i) T^*T is selfadjoint.
- (ii) $\operatorname{ran}(I + T^*T) = \mathfrak{H}$.

Proof. T^*T is automatically positive symmetric operator in the Hilbert space \mathfrak{H} , therefore Proposition 1.6 applies. \square

We have as a consequence the classical result of von Neumann [37] with essentially simpler and shorter proof:

Corollary 1.8. *Let T be densely defined and closed operator between the real or complex Hilbert spaces \mathfrak{H} and \mathfrak{K} . Then T^*T is selfadjoint.*

Proof. Closedness of T gives at once the following orthogonal decomposition to closed subspaces:

$$(1.1) \quad \mathfrak{H} \times \mathfrak{K} = \{\{v, Tv\} : v \in \operatorname{dom} T\} \oplus \{\{-T^*z, z\} : z \in \operatorname{dom} T^*\}.$$

As a consequence, for each $x \in \mathfrak{H}$ we find a (unique) pair $v \in \operatorname{dom} T$ and $z \in \operatorname{dom} T^*$ such that

$$x = v - T^*z, \quad 0 = Tv + z.$$

This yields $v \in \operatorname{dom} T^*T$ and that

$$x = (I + T^*T)v.$$

In particular, $\operatorname{ran}(I + T^*T) = \mathfrak{H}$ and thus Theorem 1.7 applies. \square

1.3. Characterization of skew-adjoint operators

Our revised version of von Neumann's characterization of selfadjoint operators on complex Hilbert space follow by the next result:

Theorem 1.9. *Let A be a not necessarily densely defined linear operator in a real or complex Hilbert space \mathfrak{H} . Then equivalent statements are:*

- (i) A is skew-adjoint.
- (ii) a) A is skew-symmetric,
 b) $\text{ran}(I + A) = \mathfrak{H}$,
 c) $\text{ran}(I - A) = \mathfrak{H}$.

Proof. Both of conditions (i) and (ii) a) imply that

$$\|x \pm Ax\|^2 = \|Ax\|^2 + \|x\|^2 \geq \|x\|^2, \quad (x \in \text{dom } A).$$

Therefore, $I \pm A$ is bounded from below in each cases.

Assume first that A is skew-adjoint. Then A is densely defined and closed so that the operators $I \pm A$ have closed ranges. Thus we have $\text{ran}(I \pm A) = \mathfrak{H}$ thanks to the following identities

$$\{0\} = \ker(I \mp A) = \ker(I \pm A)^* = \{\text{ran}(I \pm A)\}^\perp.$$

Conversely, let us assume (ii) and prove (i). First of all A is automatically densely defined: for if $z \in \{\text{dom } A\}^\perp$ then $z = y + Ay$ for some $y \in \text{dom } A$, thus

$$0 = (x, z) = (x, y + Ay) = ((I - A)x, y), \quad (x \in \text{dom } A)$$

follows. Therefore, $y \in \{\text{ran}(I - A)\}^\perp = \{0\}$ and thus $z = 0$, indeed. As a consequence, A^* exists and extends $-A$ by assumption (ii) a) so that we only have to check $\text{dom } A^* \subseteq \text{dom } A$. Let $z \in \text{dom } A^*$ and take y from $\text{dom } A$ such that $z - A^*z = y + Ay$. Then $y + Ay = y - A^*y$ also satisfies so that

$$(z - y) \in \ker(I - A^*) = \ker(I - A)^* = \{\text{ran}(I - A)\}^\perp = \{0\}.$$

Therefore $z = y \in \text{dom } A$ follows, as desired. □

Theorem 1.10. *Let B be a not necessarily densely defined operator in a complex Hilbert space \mathfrak{H} . The following statements are equivalent:*

- (i) B is selfadjoint.
- (ii) a) B is symmetric,
 b) $\text{ran}(iI + B) = \mathfrak{H}$,
 c) $\text{ran}(iI - B) = \mathfrak{H}$.

Proof. Apply Theorem 1.9 for the operator $A = iB$. □

From the above theorem the following classical result easily follows.

Corollary 1.11. *Let (X, \mathfrak{M}, μ) be a measure space. Then the multiplication by a real valued measurable function f on the complex Hilbert space $\mathcal{L}_C^2(X, \mathfrak{M}, \mu)$ is selfadjoint.*

Proof. For a given h from $\mathcal{L}_C^2(X, \mathfrak{M}, \mu)$ we find that

$$\left| \frac{h}{f \pm i} \right|^2 = \frac{|h|^2}{|f|^2 + 1} \leq |h|^2, \quad \left| \frac{f \cdot h}{f \pm i} \right|^2 = |h|^2 \cdot \frac{|f|^2}{|f|^2 + 1} \leq |h|^2,$$

therefore that $h \cdot (f \pm i)^{-1}$ is in the maximal domain in $\mathcal{L}_C^2(X, \mathfrak{M}, \mu)$ of the multiplication operator by the function f . Theorem 1.10 therefore applies. \square

Note that the usual proofs of the above statement are going to prove first that the multiplication operator in question has dense domain. In our context the denseness of the domain is automatically obtained.

Our next result is a part of the classical theorem of M. H. Stone [59] in the complex Hilbert space case.

Corollary 1.12. *Let $U : \mathbb{R} \rightarrow \mathcal{B}(\mathfrak{H})$ be strongly continuous one parameter unitary group representation of the real numbers over the real (or complex) Hilbert space \mathfrak{H} . The derivative A of U at zero given by*

$$\text{dom } A := \left\{ x \in \mathfrak{H} : \lim_{t \rightarrow 0} \frac{U(t)x - x}{t} \text{ exists} \right\}, \quad Ax := \lim_{t \rightarrow 0} \frac{U(t)x - x}{t}$$

is skew-adjoint operator in \mathfrak{H} .

Proof. For any $x \in \mathfrak{H}$ one easily obtains that

$$x_+ := \int_0^{+\infty} e^{-s} U(s)x \, ds, \quad x_- := \int_{-\infty}^0 e^s U(s)x \, ds$$

as absolute convergent improper Riemann integrals exist and satisfy

$$x = (I - A)x_+, \quad x = (I + A)x_-.$$

Consequently, Theorem 1.9 applies, see [64] for the details. \square

Note that iA is selfadjoint operator if \mathfrak{H} is complex Hilbert space (see the classical Stone's Theorem [58]).

1.4. Further characterizations

Full range property on the product space of an appropriate square matrix operator characterizes the selfadjointness of a not necessarily densely defined symmetric operator:

Theorem 1.13. *Let A be a not necessarily densely defined operator in a real or complex Hilbert space \mathfrak{H} . The following statements are equivalent:*

- (i) A is selfadjoint.
- (ii) a) A is symmetric,

$$\text{b) } \operatorname{ran} \begin{pmatrix} A & I \\ -I & A \end{pmatrix} = \mathfrak{H} \times \mathfrak{H}.$$

Proof. In each cases we have the following identity:

$$\begin{aligned} \left\| \begin{pmatrix} A & -I \\ I & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 &= \|Ax - y\|^2 + \|x + Ay\|^2 \\ &= \|Ax\|^2 + \|Ay\|^2 + \|x\|^2 + \|y\|^2 \\ &\geq \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2, \end{aligned}$$

for all $x, y \in \operatorname{dom} A$. Consequently, the operator matrix $\begin{pmatrix} A & I \\ -I & A \end{pmatrix}$ is bounded from below. Secondly, for each $u, v \in \mathfrak{H}$ we find $x, y \in \operatorname{dom} A$ such that

$$\begin{pmatrix} A & I \\ -I & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ -v \end{pmatrix}.$$

Therefore we have

$$\begin{pmatrix} A & -I \\ I & A \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

This means that $\operatorname{ran} \begin{pmatrix} A & I \\ -I & A \end{pmatrix} = \mathfrak{H} \times \mathfrak{H}$ automatically implies that $\operatorname{ran} \begin{pmatrix} A & -I \\ I & A \end{pmatrix} = \mathfrak{H} \times \mathfrak{H}$. Now check that A is automatically densely defined: Take $z \in \{\operatorname{dom} A\}^\perp$, then for some $u, v \in \operatorname{dom} A$ we have $\begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} A & I \\ -I & A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$. Consequently, for all $x, y \in \operatorname{dom} A$

$$0 = \left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ 0 \end{pmatrix} \right) = \left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} A & I \\ -I & A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right) = \left(\begin{pmatrix} A & -I \\ I & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right),$$

therefore $\begin{pmatrix} u \\ v \end{pmatrix} \in \left\{ \operatorname{ran} \begin{pmatrix} A & -I \\ I & A \end{pmatrix} \right\}^\perp = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$, thus $z = 0$.

All in all A^* exists and extends A so that we should only check that $\operatorname{dom} A^* \subseteq \operatorname{dom} A$. If $z \in \operatorname{dom} A^*$ then for some $u, v \in \operatorname{dom} A$ we have that

$$\begin{pmatrix} A^* & I \\ -I & A^* \end{pmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} A & I \\ -I & A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A^* & I \\ -I & A^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Consequently,

$$\begin{pmatrix} z - u \\ 0 - v \end{pmatrix} \in \ker \begin{pmatrix} A^* & I \\ -I & A^* \end{pmatrix} = \ker \begin{pmatrix} A & -I \\ I & A \end{pmatrix}^* = \left\{ \operatorname{ran} \begin{pmatrix} A & -I \\ I & A \end{pmatrix} \right\}^\perp = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\},$$

which implies $z = u \in \operatorname{dom} A$, as desired. \square

Remark 1.14. A very same proof shows that the statement of the above theorem remains true if we replace the operator matrix in ii) b) by $T := \begin{pmatrix} A & bI \\ -bI & A \end{pmatrix}$ where $b \in \mathbb{R}$, $b \neq 0$ is any constant. Moreover, T has bounded inverse whenever A is selfadjoint.

Now, the real counterpart of Corollary 7.11 can easily be obtained:

Corollary 1.15. *The multiplication by a measurable function f over the real Hilbert space $\mathcal{L}_{\mathbb{R}}^2(X, \mathfrak{M}, \mu)$ is selfadjoint.*

Proof. Let \mathfrak{H} stand for the Hilbert space $\mathcal{L}_{\mathbb{R}}^2(X, \mathfrak{M}, \mu)$. Given $u, v \in \mathfrak{H}$ we find that the functions

$$x := \frac{f}{1+f^2}u - \frac{1}{1+f^2}v, \quad y := \frac{f}{1+f^2}v - \frac{1}{1+f^2}u$$

are from the maximal domain of the multiplication operator by f over \mathfrak{H} such that

$$f \cdot x + y = u, \quad -x + f \cdot y = v.$$

Therefore Theorem 1.13 applies. \square

As a (maybe worth mentioning) corollary of Theorem 1.13 we obtain the following characterization of orthogonal projections:

Theorem 1.16. *Let A be a symmetric operator in a real or complex Hilbert space \mathfrak{H} with $A^2 \subset A$. The following statements are equivalent:*

- (i) *A is an orthogonal projection, i.e. $A \in \mathcal{B}(\mathfrak{H})$ and $A = A^* = A^2$.*
- (ii) $\operatorname{ran} \begin{pmatrix} A & bI \\ -bI & A \end{pmatrix} = \mathfrak{H} \times \mathfrak{H}$ *for some $b \in \mathbb{R}$.*

Proof. If $b = 0$ then clearly $\operatorname{ran} A = \mathfrak{H}$ so that A is selfadjoint due to Proposition 6.2; otherwise, i.e. when $b \neq 0$, Theorem 1.13 and Remark 1.14 lead to the same conclusion. Since the selfadjoint operator A^2 does not have any proper selfadjoint extension we conclude that

$$A = A^* = A^2 = AA^*.$$

On the other hand, for any densely defined closed operator T between Hilbert spaces \mathfrak{H} and \mathfrak{K} the following identity due to (1.1) holds:

$$\mathfrak{H} = \operatorname{dom} T + \operatorname{ran} T^*.$$

Since in our case $\operatorname{ran} A^* \subseteq \operatorname{dom} A$ also satisfies, this means that $\operatorname{dom} A = \mathfrak{H}$, i.e. A is continuous according to the Banach closed graph theorem. \square

Corollary 1.17. *Let A be a symmetric operator in a real or complex Hilbert space \mathfrak{H} with $A^2 \subset A$. The following assertions are equivalent:*

- (i) $A = I$.
- (ii) A is of full range, i.e. $\operatorname{ran} A = \mathfrak{H}$.

Proof. The preceding theorem with $b = 0$ applies. \square

We close this chapter with the classical Kato-Rellich perturbation theorem [42]. In the original version of this result the underlying Hilbert space is assumed to be complex and all of the proofs are based on Theorem 1.10 of von Neumann and thus cannot be applied in the real Hilbert space case. K. Gustafson in [20] has shown that the statement of the cited theorem remains true even if the underlying space is real; his proof is given in fact after a work by D. K. Rao.

Gustafson wrote: "There are no doubt shorter and sharper versions of the proof given above ... for real Hilbert space, perhaps also a proof by complexifying and using the complex version. The latter was not evident to us at the time and in any case the proof given above holds as well for the complex case. We did not investigate the above proof in any generality (e.g., Banach space, nonlinear versions)."

Via Theorem 1.13 we are allowed to discuss the real and the complex variants together:

Theorem 1.18. *Let A be a selfadjoint operator in a real or complex Hilbert space \mathfrak{H} . Another symmetric operator B in \mathfrak{H} let be given which is A -bounded in the regular perturbation sense: $\text{dom } A \subseteq \text{dom } B$ and there are two nonnegative constants $a, b \geq 0$, $a < 1$ such that*

$$(1.2) \quad \|Bh\|^2 \leq a(\|Ah\|^2 + b^2\|h\|^2),$$

holds for all $h \in \text{dom } A$. Then $A + B$ is selfadjoint.

Proof. Since $A + B$ is symmetric, in view of Theorem 1.13 and by using the notations

$$S := \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}, \quad T := \begin{pmatrix} A & bI \\ -bI & A \end{pmatrix},$$

it suffices to show that the following identity holds:

$$(1.3) \quad \text{ran}(S + T) = \mathfrak{H} \times \mathfrak{H}.$$

Theorem 1.13 combined with Remark 1.14 shows that T is a one-to-one closed operator with full-range, so that its inverse T^{-1} acts as a bounded operator on $\mathfrak{H} \times \mathfrak{H}$. We state that ST^{-1} is bounded too with norm less than 1. Indeed, for all $g, h \in \text{dom } A$ we have

$$\begin{aligned} \left\| ST^{-1} \begin{pmatrix} Ah - bg \\ bh + Ag \end{pmatrix} \right\|^2 &= \|Bh\|^2 + \|Bg\|^2 \\ &\leq a(\|Ah\|^2 + b^2\|h\|^2 + \|Ag\|^2 + b^2\|g\|^2) \\ &= a(\|Th - bg\|^2 + \|bh + Tg\|^2) \\ &= a \left\| \begin{pmatrix} Ah - bg \\ bh + Ag \end{pmatrix} \right\|^2, \end{aligned}$$

that is $\|ST^{-1}\| \leq \sqrt{a} < 1$. Consequently, ST^{-1} does not contain -1 as spectrum point, therefore the identity

$$S + T = (ST^{-1} + I)T$$

shows that $S + T$ is of full range, i.e. $A + B$ is selfadjoint indeed. \square

CHAPTER 2

Characterization of essentially selfadjoint operators

Operators between real or complex Hilbert spaces A, B, \dots are assumed to be linear, but not necessarily densely defined satisfying one of the following two identities:

$$(2.1) \quad (Ax, y) \mp (x, Ay) = 0 \quad (x, y \in \operatorname{dom} A).$$

According to these identities the operator A is called *symmetric/skew-symmetric*, respectively.

In case when A is densely defined, i.e. its adjoint A^* exists, then the maximal requirement of the former size is

$$(2.2) \quad A = \pm A^*$$

and is called *selfadjoint* or *skew-adjoint*, respectively. A symmetric/skew-symmetric operator A on a real or complex Hilbert space is called *essentially selfadjoint/skew-adjoint* if its adjoint A^* exists (i.e. A is densely defined) and A^* is selfadjoint/skew-adjoint, respectively. In other words, A^* fulfills property (2.2):

$$(2.3) \quad A^* = \pm A^{**},$$

where A^{**} is nothing else than the closure of A . In what follows we give a simple range-characterization for the case just mentioned.

Theorem 2.1. *Let A be densely defined symmetric/skew-symmetric operator on a real or complex Hilbert space \mathfrak{H} . Then the following statements are equivalent:*

- (i) A is essentially selfadjoint/skew-adjoint;
- (ii) $\operatorname{ran} A^* = \operatorname{ran} A^{**}$;
- (iii) $\pm A^* \subset A^{**}$, i.e. A^* is symmetric/skew-symmetric.

Proof. Of course, (ii) is an evident consequence of (i). To see that (ii) implies (i) note first that A^* immediately extends $\pm A$ according to our assumption (2.1), therefore A^* also extends $\pm A^{**}$. Therefore we should only check that A^* and A^{**} have common domain of definition. Assume therefore that z belongs to $\operatorname{dom} A^*$ so that for some $y \in \operatorname{dom} A^{**}$ we have by our assumptions

$$A^*z = A^{**}(\pm y) = \pm A^*(\pm y) = A^*y.$$

Consequently,

$$(z - y) \in \ker A^* = \{\operatorname{ran} A\}^\perp = \{\operatorname{ran} A^{**}\}^\perp = \{\operatorname{ran} A^*\}^\perp = \ker A^{**},$$

and therefore

$$z = (z - y) + y \in \operatorname{dom} A^{**},$$

indeed. This chain of identities is an analogous consequence stated in Arens [4, (2.0)]. Analogously, (i) implies (iii) evidently so that we need to check that (i) is a consequence of (iii): since we have seen that A^* extends $\pm A^{**}$ in the corresponding cases, assumption (iii) immediately gives property (2.3), i.e. A is essentially selfadjoint/skew-adjoint as stated in (i). \square

Remark 2.2. The equivalence of (i) and (iii) is given in [30, Problem 3.10] of T. Kato and applied by Stone in [58, Theorem 3.4]. Kato noticed (p. 270) according to the essential selfadjointness problem: It is in general a rather complicated task to decide whether or not this is true for a given matrix $(\tau_{j,k})_{j,k \in \mathbb{N}}$.

Corollary 2.3. *Let $(\tau_{j,k})_{j,k \in \mathbb{N}}$ be a symmetric matrix:*

$$(2.4) \quad \overline{\tau_{j,k}} = \tau_{k,j} \quad (j, k = 1, 2, \dots)$$

and assume that

$$(2.5) \quad \sum_{k=1}^{\infty} |\tau_{j,k}|^2 < \infty \quad \text{for all } j \in \mathbb{N}.$$

We associate a symmetric operator in ℓ^2 with domain to be the linear manifold \mathcal{D} spanned by the canonical basis $e_n = (\delta_{k,n})_{k \in \mathbb{N}}$. Then the following statements are equivalent:

- (i) A is essentially selfadjoint;
- (ii) A^* is symmetric;
- (iii) The following linear manifolds

$$\begin{cases} \mathcal{R}_* := \{y \in \ell^2 : \sup\{|(x, y)|^2 : x \in \mathcal{D}, (Ax, Ax) \leq 1\} < \infty\}, \\ \mathcal{R}_{**} := \{z \in \ell^2 : \exists \{x_n\}_{n=1}^{\infty} \subset \mathcal{D} \text{ such that } Ax_n \rightarrow z, (x_n - x_m) \rightarrow 0\} \end{cases}$$

are equal.

Proof. The sets \mathcal{R}_* and \mathcal{R}_{**} in condition (iii) are identified as $\operatorname{ran} A^*$ and $\operatorname{ran} A^{**}$, respectively according to [44, Theorem 1] and due to the fact that A^{**} is just the closure of A . Theorem 2.1 therefore applies and we are done. \square

An easy range assumption appears in the following statement:

Theorem 2.4. *Let $A : \mathfrak{H} \rightarrow \mathfrak{H}$ be symmetric/skew-symmetric closable operator such that its closure \overline{A} is of full range:*

$$\operatorname{ran} \overline{A} = \mathfrak{H}.$$

Then A is essentially selfadjoint/skew-adjoint.

Proof. Our first aim is to prove that A is densely defined: for $z \in \{\operatorname{dom} A\}^\perp$ we find by assumption $\{y_n\}_{n=1}^\infty$ from $\operatorname{dom} A$ such that $y := \lim_{n \rightarrow \infty} y_n$ exists and $Ay_n \rightarrow z$. Consequently, for each x from $\operatorname{dom} A$ we have

$$0 = (x, z) = \lim_{n \rightarrow \infty} (x, Ay_n) = \lim_{n \rightarrow \infty} (Ax, y_n) = (Ax, y),$$

so that $\mathfrak{H} = \operatorname{ran} \bar{A} \subseteq \overline{\operatorname{ran} A}$ implies $y \in \{\operatorname{ran} A\}^\perp = \{0\}$, i.e. $y = 0$. But then $z = 0$, by the closability of A , so that we have done.

Consequently, A^* exists and extends $\pm A$, therefore A^* extends $\pm A^{**} = \pm \bar{A}$. This means among other that A^* is of full range too, and therefore that $\operatorname{ran} A^* = \operatorname{ran} A^{**}$. Consequently, A is essentially selfadjoint/skew-adjoint due to Theorem 2.1. \square

2.1. Characterization of essentially skew/self-adjoint operators

In what follows a (skew-)symmetric operator A is given on a real or complex Hilbert space, not at all assuming that it is densely defined, but under some natural conditions it turns out to be even essentially skew/self-adjoint, that is its adjoint and closure exist and are at the same time equal.

Theorem 2.5. *Let \mathfrak{H} be real or complex Hilbert space, $A : \mathfrak{H} \rightarrow \mathfrak{H}$ be closable not necessarily densely defined and (skew-)symmetric linear operator. The following two statements are equivalent:*

- (i) A is essentially skew/self-adjoint;
- (ii) a) A is (skew-)symmetric
- b) $\{\operatorname{ran} A\}^\perp = \{x \in \mathfrak{H} : \exists \{x_n\}_{n=1}^\infty \subset \operatorname{dom} A, x_n \rightarrow x, Ax_n \rightarrow 0\}$
- c) The following linear manifolds

$$\begin{cases} \mathcal{R}_* := \{y \in \mathfrak{H} : \sup\{|(x, y)|^2 : x \in \operatorname{dom} A, (Ax, Ax) \leq 1\} < \infty\}, \\ \mathcal{R}_{**} := \{z \in \mathfrak{H} : \exists \{x_n\}_{n=1}^\infty \subset \operatorname{dom} A \text{ such that } Ax_n \rightarrow z, (x_n - x_m) \rightarrow 0\} \end{cases}$$

are equal.

Proof. It is easy to see that (i) implies (ii) since a) is evident, b) explains $\ker A^* = \ker \bar{A}$, and c) reads as follows: $\operatorname{ran} A^* = \operatorname{ran} \bar{A}$; at the same time we assume $A^* = \bar{A}$.

Now assume (ii) and prove (i): first of all we check that A is automatically densely defined. Assume that $z \in \{\operatorname{dom} A\}^\perp$, i.e. that

$$(x, z) = 0 \quad (x \in \operatorname{dom} A)$$

satisfies. Then, according to (ii) c) we find $\{x_n\}_{n=1}^\infty$ from $\operatorname{dom} A$ so that $Ax_n \rightarrow z$ while $\|x_n - x_m\| \rightarrow 0$. Then, of course,

$$0 = (x, z) = \lim_{n \rightarrow \infty} (x, Ax_n) = \pm \lim_{n \rightarrow \infty} (Ax, x_n) = \pm (Ax, \lim_{n \rightarrow \infty} x_n),$$

that is, $\lim_{n \rightarrow \infty} x_n \in \{\text{ran } A\}^\perp$, therefore there exists $\{u_n\}_{n=1}^\infty$ from $\text{dom } A$ such that

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} x_n \quad \text{and} \quad Au_n \rightarrow 0.$$

In any case we have

$$x_n - u_n \rightarrow 0, \quad A(x_n - u_n) \rightarrow z,$$

and therefore by closability assumption $z = 0$ follows.

We have seen that A is automatically densely defined (skew-)symmetric operator, i.e. $A \subset \pm A^*$. Therefore,

$$A \subset A^{**} = \overline{A} \subset A^*$$

as well. But

$$\ker A^* = \{\text{ran } A\}^\perp = \ker \overline{A},$$

according to (ii) b), and $\text{ran } A^* = \text{ran } \overline{A}$ by assumption (ii) c). Therefore, $A^* = \pm \overline{A}$ follows according to Theorem 2.1. \square

2.2. Characterization of essentially skew-adjoint operators on real or complex Hilbert space

We will prove in what follows that an essentially skew-adjoint operator on real or complex Hilbert space is basically simple to characterize among the non-densely defined skew-symmetric closable operators by the Neumann type condition.

Theorem 2.6. *Let \mathfrak{H} be real or complex Hilbert space and A be a linear skew-symmetric closable operator not assumed to be densely defined. The following statements are equivalent:*

- (i) $A^* = -\overline{A}$, i.e. A is densely defined and essentially skew-adjoint;
- (ii) $\{\text{ran}(I + A)\}^\perp = \{\text{ran}(I - A)\}^\perp = \{0\}$.

Proof. First of all, by the skew-symmetry of A we have the following identity

$$\|(I \pm A)x\|^2 = \|x\|^2 + \|Ax\|^2 \quad (x \in \text{dom } A).$$

Therefore, by assuming (i), it follows that

$$\{\text{ran}(I \pm A)\}^\perp = \ker(I \pm A)^* = \ker(I \pm A^*) = \ker(I \mp \overline{A}) = \ker(I \mp A) = \{0\},$$

since $I \pm A$ are bounded below by 1. This is stated in (ii) exactly. We proceed by the proof of (ii) implies (i). First of all we show that A is automatically densely defined: for if $z \in \{\text{dom } A\}^\perp$ then by assumption there is $\{u_n\}_{n=1}^\infty$ from $\text{dom } A$ such that $u := \lim_{n \rightarrow \infty} u_n$ exists and $u_n + Au_n \rightarrow z$. Therefore, $Au_n \rightarrow (z - u)$ so that

$$0 = (x, z) = \lim_{n \rightarrow \infty} (x, u_n + Au_n) = \lim_{n \rightarrow \infty} (x, u_n) - \lim_{n \rightarrow \infty} (Ax, u_n) = (x - Ax, u)$$

holds for each $x \in \text{dom } A$. Consequently, $u \in \{\text{ran}(I - A)\}^\perp = \{0\}$, i.e. $u = 0$, and by the closability of A we have $0 = \lim_{n \rightarrow \infty} Au_n = z - u$, that is $z = u = 0$, as it is claimed.

It then follow by the skew-symmetry of A that $-\overline{A} = -A^{**} \subset A^*$, and thus that the proof is complete by showing $\operatorname{dom} A^* = \operatorname{dom} \overline{A}$. If $z \in \operatorname{dom} A^*$ then

$$z - A^*z = \lim_{n \rightarrow \infty} (v_n + Av_n)$$

holds for some $\{v_n\}_{n=1}^\infty$ from $\operatorname{dom} A$, where $v = \lim_{n \rightarrow \infty} v_n$ exists. Then, of course,

$$Av_n \rightarrow (z - v) - A^*z.$$

But here $v \in \operatorname{dom} \overline{A}$, therefore we have that

$$\overline{A}v = (z - v) - A^*z.$$

Consequently, $\overline{A}v = -A^*v$ implies

$$(z - v) \in \ker(I - A^*) = \ker(I - A)^* = \{\operatorname{ran}(I - A)\}^\perp = \{0\},$$

therefore that $z = v \in \operatorname{dom} \overline{A}$ holds true, indeed. \square

As a consequence, we have the revised form of von Neumann's characterization of essentially selfadjoint operators on *complex* Hilbert spaces (see [36]), namely we need not assume, beside the range assumption, the existence of the adjoint, only a weaker closability assumption.

Corollary 2.7. *Let \mathfrak{H} be a complex Hilbert space, $A : \mathfrak{H} \rightarrow \mathfrak{H}$ be a symmetric, not necessarily densely defined linear operator. The following two statements are equivalent:*

- (i) $A^* = \overline{A}$, i.e. A is essentially selfadjoint;
- (ii) $\{\operatorname{ran}(iI + A)\}^\perp = \{0\} = \{\operatorname{ran}(iI - A)\}^\perp$

Proof. The operator iA fulfills assumption (ii) of the former Theorem 2.6. \square

2.3. Characterization of essentially selfadjoint positive operators

Throughout this section we assume that A is positive symmetric operator on a real or complex Hilbert space \mathfrak{H} in the classical sense that

$$(2.6) \quad (Ax, x) \geq 0 \quad (x \in \operatorname{dom} A)$$

holds true. Note that we do not assume that the operator A is densely defined, only that the operator A is symmetric (what is automatic if the space is complex).

Theorem 2.8. *Let A be a positive symmetric closable linear operator on the real or complex Hilbert space \mathfrak{H} . Then equivalent statements are:*

- (i) A is densely defined and essentially selfadjoint;
- (ii) $\{\operatorname{ran}(I + A)\}^\perp = \{0\}$.

Proof. The positive symmetricity of A has the following useful relation:

$$\|(I + A)x\|^2 = \|x\|^2 + \|Ax\|^2 + 2(Ax, x) \geq \|x\|^2 \quad (x \in \operatorname{dom} A).$$

Of course, by closability (and the bounded below property of A) implies that

$$\operatorname{ran}(\overline{I + A}) = \operatorname{ran}(I + \overline{A}) = \overline{\operatorname{ran}(I + A)} = \mathfrak{H}$$

holds in both cases, i.e. (i) clearly implies (ii). To prove the reverse implication assume (ii). First of all A is densely defined: for if $z \in \{\operatorname{dom} A\}^\perp$, we have on the one hand

$$(x, z) = 0 \quad (x \in \operatorname{dom} A).$$

On the other hand there exists $\{w_n\}_{n=1}^\infty$ from $\operatorname{dom} A$ such that $(w_n + Aw_n) \rightarrow z$, while $w_n \rightarrow w$. Therefore, $Aw_n \rightarrow (z - w)$ so that $\overline{A}w = z - w$. But then for each x from $\operatorname{dom} A$ we have

$$0 = (x, z) = \lim_{n \rightarrow \infty} (x, w_n + Aw_n) = (x, w) + (Ax, w) = ((I + A)x, w).$$

This means, of course, that $w \in \{\operatorname{ran}(I + A)\}^\perp = \{0\}$, that is $w = 0$ and then, by closability of A , $(z - w) = 0$, i.e. $z = w = 0$, indeed.

Then of course, $A \subset A^*$ by assumption and therefore $A^{**} \subset A^*$ and to close the proof we need only that $\operatorname{dom} A^* = \operatorname{dom} \overline{A}$. For $z \in \operatorname{dom} A^*$ we find $\{u_n\}_{n=1}^\infty$ from $\operatorname{dom} A$ such that $u_n + Au_n \rightarrow z + A^*z$, while $u_n \rightarrow u$. This means that $u \in \operatorname{dom} \overline{A}$ and $A^*u = \overline{A}u = z + A^*z - u$. But then $(I + A^*)(z - u) = 0$, that is

$$(z - u) \in \ker(I + A^*) = \ker(I + A)^* = \{\operatorname{ran}(I + A)\}^\perp = \{0\}.$$

Consequently, $z = u \in \operatorname{dom} \overline{A}$, indeed. \square

Corollary 2.9. *Let $T : \mathfrak{H} \rightarrow \mathfrak{K}$ be densely defined linear operator between real or complex Hilbert spaces. Then equivalent statements are:*

- (i) T^*T is essentially selfadjoint;
- (ii) $\{\operatorname{ran}(I + T^*T)\}^\perp = \{0\}$.

Proof. T^*T is positive symmetric operator in the Hilbert space \mathfrak{H} , and surely selfadjoint if T is closed, thanks to von Neumann (see [37]). In general, its domain of definition is not dense (may be trivial), therefore it is not obvious that T^*T is closable at all. Our only claim is therefore to prove the closability of T^*T : for if $\{x_n\}_{n=1}^\infty$ from $\operatorname{dom} T^*T$ we assume $x_n \rightarrow 0$ and $T^*Tx_n \rightarrow y$, then for all $x \in \operatorname{dom} A$ we have

$$(x, y) = \lim_{n \rightarrow \infty} (x, T^*Tx_n) = \lim_{n \rightarrow \infty} (Tx, Tx_n),$$

where

$$|(Tx, Tx_n)|^2 \leq (Tx, Tx)(Tx_n, Tx_n) = (Tx, Tx)(x_n, T^*Tx_n),$$

such that $(x_n, T^*Tx_n) \rightarrow (0, y) = 0$. Therefore, $y \in \{\operatorname{dom} T\}^\perp = \{0\}$, that is $y = 0$, indeed. This means that T^*T is closable, so that Theorem 2.8 applies. \square

Remark 2.10. The above T^*T always has positive selfadjoint extension as proved in [53, Theorem 2.1].

2.4. Characterization of essentially selfadjoint operators

Let \mathfrak{H} be real or complex Hilbert space, $A : \mathfrak{H} \rightarrow \mathfrak{H}$ be a closable linear operator whose domain of definition is not assumed to be dense. We give in this section necessary and sufficient conditions in terms of dense range property of an appropriate square matrix operator in the product space $\mathfrak{H} \times \mathfrak{H}$:

Theorem 2.11. *Let \mathfrak{H} be real or complex Hilbert space, $A : \mathfrak{H} \rightarrow \mathfrak{H}$ not necessarily densely defined operator. The following statements are equivalent:*

- (i) A is densely defined and essentially selfadjoint;
- (ii) a) A is closable;
- b) A is symmetric;
- c) $\text{ran} \begin{pmatrix} A & I \\ -I & A \end{pmatrix}$ is dense in $\mathfrak{H} \times \mathfrak{H}$.

Proof. In both cases for all $u, v \in \text{dom } A$ the following satisfies

$$(2.7) \quad \left\| \begin{pmatrix} A & \pm I \\ \mp I & A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2 = \|Au\|^2 + \|Av\|^2 + \|u\|^2 + \|v\|^2 \geq \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2$$

Consequently if A is essentially selfadjoint according to (i), then \bar{A} exists as a selfadjoint operator in \mathfrak{H} , therefore the operators $\begin{pmatrix} \bar{A} & \pm I \\ \mp I & \bar{A} \end{pmatrix}$ are also closed, bounded below, and thus their range are $\mathfrak{H} \times \mathfrak{H}$, since they are adjoints of each other. We need only to show that (ii) implies (i). We proceed just the same way as before, i.e. we prove first that A is densely defined: for $z \in \{\text{dom } A\}^\perp$ we prove $z = 0$. By our assumption there exist $\{u_n\}_{n=1}^\infty, \{v_n\}_{n=1}^\infty$ from $\text{dom } A$ such that $\begin{pmatrix} A & I \\ -I & A \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \rightarrow \begin{pmatrix} z \\ 0 \end{pmatrix}$, and therefore for $x, y \in \text{dom } A$ we find

$$0 = \left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ 0 \end{pmatrix} \right) = \lim_{n \rightarrow \infty} \left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} A & I \\ -I & A \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right) = \lim_{n \rightarrow \infty} \left(\begin{pmatrix} A & -I \\ I & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right).$$

But here, $Au_n + v_n \rightarrow z$, and $-u_n + Av_n \rightarrow 0$ and, thanks to (4.19), $u_n \rightarrow u$ and $v_n \rightarrow v$ holds true for some u and v . It implies that for $x, y \in \text{dom } A$

$$0 = \left(\begin{pmatrix} A & -I \\ I & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right)$$

satisfies. This means exactly that

$$0 = \left(\begin{pmatrix} A & I \\ -I & A \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix}, \begin{pmatrix} u \\ -v \end{pmatrix} \right)$$

holds also true for each $x, y \in \text{dom } A$, that is $\begin{pmatrix} u \\ -v \end{pmatrix}$ belongs to the orthocomplement of range of the operator $\begin{pmatrix} A & I \\ -I & A \end{pmatrix}$, i.e. equals $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This implies $u = v = 0$, so that $z = \lim_{n \rightarrow \infty} Au_n + \lim_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} Au_n$, therefore that $\lim_{n \rightarrow \infty} Au_n = 0$, i.e. $z = 0$ by closability of A .

Now, since A^* exists, A^* extends A according to the symmetry of A , and also extends the closure \overline{A} of A . To check that they actually are the same, needs only to show that their domain of definitions are equal. Let therefore $z \in \text{dom } A^*$, and prove $z \in \text{dom } \overline{A}$. Once more there exists $\{u_n\}_{n=1}^\infty, \{v_n\}_{n=1}^\infty$ from $\text{dom } A$ such that

$$\begin{pmatrix} A^* & I \\ -I & A^* \end{pmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix} = \lim_{n \rightarrow \infty} \begin{pmatrix} A & I \\ -I & A \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \lim_{n \rightarrow \infty} \begin{pmatrix} A^* & I \\ -I & A^* \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix},$$

i.e. $Au_n + v_n \rightarrow A^*z$ and $-u_n + A^*v_n \rightarrow -z$, where, of course, $u_n \rightarrow u$ and $v_n \rightarrow v$ for some $u, v \in \mathfrak{H}$. Therefore, $u, v \in \text{dom } \overline{A}$ and $\overline{A}u = A^*z - v$, $\overline{A}v = u - z$, as well. Consequently,

$$A^*u_n \rightarrow A^*z - v, \quad A^*v_n \rightarrow u - z.$$

But then, in account of the closedness of A^*

$$A^*u = A^*z - v, \quad A^*v = u - z,$$

what follows

$$\begin{pmatrix} A^* & I \\ -I & A^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A^* & I \\ -I & A^* \end{pmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix}.$$

This exactly means that

$$\begin{pmatrix} z - u \\ -v \end{pmatrix} \in \ker \begin{pmatrix} A^* & I \\ -I & A^* \end{pmatrix} = \left\{ \text{ran} \begin{pmatrix} A & I \\ -I & A \end{pmatrix} \right\}^\perp = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e. that $z = u \in \text{dom } \overline{A}$, $v \in \text{dom } \overline{A}$ at the same time, and we have done. \square

2.5. Theorem of Wüst in real or complex Hilbert space

As an application of Theorem 2.11 we present a new proof of the classical Wüst perturbation theorem by discussing simultaneously the real and the complex cases:

Theorem 2.12. *Let T be a selfadjoint operator in a real or complex Hilbert space \mathfrak{H} . If S is a T -bounded symmetric operator by T bound 1, i.e. there exists a constant $b > 0$ such that*

$$\|Sh\|^2 \leq \|Th\|^2 + b^2\|h\|^2, \quad (h \in \text{dom } T),$$

then $T + S$ is essentially selfadjoint.

Proof. Let $\{t_n\}_{n=1}^\infty$ be a sequence with $0 < t_n < 1$ for $n \in \mathbb{N}$ tending to 1. Let us introduce the following notations:

$$A_n := \begin{pmatrix} T + t_n S & -bI \\ bI & T + t_n S \end{pmatrix}, \quad A := \begin{pmatrix} T + S & -bI \\ bI & T + S \end{pmatrix}.$$

We prove that $\text{ran } A$ is dense in $\mathfrak{H} \times \mathfrak{H}$. To obtain that we conclude first that for any $f, g \in \text{dom } T$ the following inequality holds:

$$(2.8) \quad \left\| (A - A_n) \begin{pmatrix} f \\ g \end{pmatrix} \right\| \leq \left\| A_n \begin{pmatrix} f \\ g \end{pmatrix} \right\|.$$

Indeed, by using the fact that S is T -bounded by T -bound 1, one obtains that

$$\begin{aligned} \left\| (A - A_n) \begin{pmatrix} f \\ g \end{pmatrix} \right\| &= (1 - t_n) \left\| \begin{pmatrix} Sf \\ Sg \end{pmatrix} \right\| = (1 - t_n) \sqrt{\|Sf\|^2 + \|Sg\|^2} \\ &\leq \sqrt{\|Tf\|^2 + b^2\|f\|^2 + \|Tg\|^2 + b^2\|g\|^2} - t_n \sqrt{\|Sf\|^2 + \|Sg\|^2} \\ &= \left\| \begin{pmatrix} Tf - bg \\ bf + Tg \end{pmatrix} \right\| - t_n \left\| \begin{pmatrix} Sf \\ Sg \end{pmatrix} \right\| \\ &\leq \left\| \begin{pmatrix} Tf + t_n Sf - bg \\ bf + Tg + t_n Sg \end{pmatrix} \right\| \\ &= \left\| A_n \begin{pmatrix} f \\ g \end{pmatrix} \right\|. \end{aligned}$$

In order to prove that $\text{ran } A$ is dense, let $h, k \in \mathfrak{H}$ such that

$$\begin{pmatrix} h \\ k \end{pmatrix} \in \{\text{ran } A\}^\perp.$$

According to the Kato–Rellich theorem (see Theorem 1.18 above), for each $n \in \mathbb{N}$ the operator $T + t_n S$ is selfadjoint, therefore operator matrix A_n is of full range and thus, in particular, for each $n \in \mathbb{N}$ there are two vectors $f_n, g_n \in \text{dom } S$ such that

$$\begin{pmatrix} h \\ k \end{pmatrix} = A_n \begin{pmatrix} f_n \\ g_n \end{pmatrix}.$$

We note that the above identity together with (2.8) implies the following inequalities:

$$(2.9) \quad \left\| \begin{pmatrix} f_n \\ g_n \end{pmatrix} \right\| \leq \frac{1}{b} \left\| A_n \begin{pmatrix} f_n \\ g_n \end{pmatrix} \right\| = \frac{1}{b} \left\| \begin{pmatrix} h \\ k \end{pmatrix} \right\|,$$

$$(2.10) \quad \left\| (A - A_n) \begin{pmatrix} f_n \\ g_n \end{pmatrix} \right\| \leq \left\| A_n \begin{pmatrix} f_n \\ g_n \end{pmatrix} \right\| = \left\| \begin{pmatrix} h \\ k \end{pmatrix} \right\|.$$

Finally, since $\text{dom } T$ is dense in \mathfrak{H} , for any $\varepsilon > 0$ there is a pair $h_\varepsilon, k_\varepsilon \in \text{dom } T$ such that $\|h - h_\varepsilon\|^2 + \|k - k_\varepsilon\|^2 \leq \varepsilon^2$. Thus, according to (2.9) and (2.10), we conclude that

$$\begin{aligned} \left\| \begin{pmatrix} h \\ k \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} h \\ k \end{pmatrix}, A_n \begin{pmatrix} f_n \\ g_n \end{pmatrix} \right\rangle = \left\| \begin{pmatrix} h \\ k \end{pmatrix}, (A_n - A) \begin{pmatrix} f_n \\ g_n \end{pmatrix} \right\rangle \\ &\leq \left\| \begin{pmatrix} h - h_\varepsilon \\ k - k_\varepsilon \end{pmatrix} \right\| \left\| (A_n - A) \begin{pmatrix} f_n \\ g_n \end{pmatrix} \right\| + \left\| (A_n - A) \begin{pmatrix} h_\varepsilon \\ k_\varepsilon \end{pmatrix} \right\| \left\| \begin{pmatrix} f_n \\ g_n \end{pmatrix} \right\| \\ &\leq \varepsilon \left\| \begin{pmatrix} h \\ k \end{pmatrix} \right\| + \frac{(1 - t_n)}{b} \left\| \begin{pmatrix} S h_\varepsilon \\ S k_\varepsilon \end{pmatrix} \right\| \left\| \begin{pmatrix} h \\ k \end{pmatrix} \right\|. \end{aligned}$$

Since $\{t_n\}_{n=1}^\infty$ tends to 1, by letting $n \rightarrow \infty$ one obtains that

$$\left\| \begin{pmatrix} h \\ k \end{pmatrix} \right\|^2 \leq \varepsilon \left\| \begin{pmatrix} h \\ k \end{pmatrix} \right\|.$$

This implies $h = k = 0$, as it is claimed. It can be proved in a completely analogous way that

$$\overline{\text{ran} \begin{pmatrix} T + S & bI \\ -bI & T + S \end{pmatrix}} = \mathfrak{H} \times \mathfrak{H},$$

so Theorem 2.11 completes the proof. \square

CHAPTER 3

T^*T always has a positive selfadjoint extension

One of the basic results of the theory of unbounded operators is due to John von Neumann [37]: if T is a densely defined closed operator between Hilbert spaces then T^*T is a positive selfadjoint operator. If we leave out the condition that T is closed, the conclusion is not true anymore. Moreover, in general T^*T is not even densely defined even has trivial domain of definition. Note that the first assumption of Neumann's theorem (i.e. the density of $\text{dom } T$) ensures that T^* exists as a not necessarily densely defined but closed operator. We notice that the adjoint of any operator can be defined in a relation sense as a "multivalued operator", see Arens [4] (also for a generalized Neumann theorem).

However, the operator T^*T is positive in the sense that

$$(T^*Tx, x) \geq 0 \quad \text{for } x \in \text{dom } T^*T,$$

so if we require additionally that $\text{dom } T^*T$ is also dense then classical results of Friedrichs [18], Krein [32], [33], Ando and Nishio [3] imply that T^*T admits its largest (Friedrichs) and smallest (Krein-von Neumann) extensions, usually denoted by $(T^*T)_F$ and $(T^*T)_N$, respectively.

In the present chapter, following and simplifying the treatment of Sebestyén and Stochel [49], we prove that T^*T always admits the Krein-von Neumann extension via von Neumann's theorem using a model Hilbert space, the range closure of the restriction of T to the domain of T^*T . The Friedrichs extension of T^*T also will be given in the only possible case when T^*T is densely defined. Moreover, we show that for each densely defined positive operator A there is a closable operator T , acting in the original Hilbert space, so that $A = T^*T$ and that T^*T^{**} equals the Friedrichs extension of A . The idea of our method is based on [39], [40] of Prokaj and Sebestyén.

3.1. The Krein-von Neumann extension of T^*T

First, we give the main result of the chapter as follows:

Theorem 3.1. *Let T be a densely defined operator between Hilbert spaces \mathfrak{H} and \mathfrak{K} . Then T^*T has a positive selfadjoint extension.*

Proof. Let S denote the restriction of T to $\text{dom } T^*T$ and \mathfrak{K}_T the Hilbert space $\overline{\text{ran } S}$. The inner product of \mathfrak{K}_T , i.e. the restriction of (\cdot, \cdot) to $\overline{\text{ran } S} \times \overline{\text{ran } S}$ will be denoted by $\langle \cdot, \cdot \rangle$.

Then the following relation

$$(3.1) \quad J : \text{ran } S \rightarrow \mathfrak{H}, \quad Sh \mapsto T^*Th$$

defines a densely defined operator from \mathfrak{K}_T into \mathfrak{H} with $\text{dom } J = \text{ran } S$. Moreover, the following inequality

$$|(T^*Tf, h)|^2 = |(Tf, Th)|^2 \leq \langle Sf, Sf \rangle \cdot \|Th\|^2$$

for $h \in \text{dom } T$ and $f \in \text{dom } T^*T$ ensures that the linear functional

$$Sf \mapsto (J(Sf), h) = (T^*Tf, h)$$

is continuous on $\text{ran } S = \text{dom } J$ by norm bound $\|Th\|$ thus implying $\text{dom } T \subseteq \text{dom } J^*$. Consequently, $\text{dom } J^*$ is dense so that J is closable. On the other hand for any $f, h \in \text{dom } S$ we find that

$$\langle J^*h - Sh, Sf \rangle = (h, T^*Tf) - (Th, Tf) = 0.$$

This implies that $J^*h - Sh = 0$ since $\text{ran } S$ is dense in \mathfrak{K}_T . Consequently, the following characteristic property of J^* holds:

$$(3.2) \quad J^*h = Sh \in \mathfrak{K}_T \quad (h \in \text{dom } T^*T).$$

Finally, thanks to von Neumann's theorem $J^{**}J^*$ is a positive selfadjoint operator which extends the original operator T^*T :

$$J^{**}J^*h = J^{**}(Sh) = J(Sh) = T^*Th$$

holds for all $h \in \text{dom } T^*T$. The proof is complete. \square

We recall the definition of the partial ordering \leq over the set of all positive selfadjoint operators in a Hilbert space, see Kato [29]: If A and B are positive selfadjoint operators then $A \leq B$ if and only if $\text{dom } B^{1/2} \subset \text{dom } A^{1/2}$ and

$$\|A^{1/2}x\|^2 \leq \|B^{1/2}x\|^2 \quad \text{for all } x \in \text{dom } B^{1/2}.$$

Below Theorem 3.3 says that the operator $J^{**}J^*$ is the smallest, with respect to this partial ordering, among all positive selfadjoint extensions of T^*T . The following lemma will be useful for the proof, also for the later uses:

Lemma 3.2. *Let T be a densely defined operator from \mathfrak{H} into \mathfrak{K} and let $J^{**}J^*$ be the positive selfadjoint extension of T^*T constructed in the proof of Theorem 3.1. Then $\text{dom}(J^{**}J^*)^{1/2} = \mathcal{D}_*$ where*

$$(3.3) \quad \mathcal{D}_* := \{h \in \mathfrak{H} : |(T^*Tf, h)|^2 \leq m_h \cdot \|Tf\|^2 \text{ for all } f \in \text{dom } T^*T\}$$

and for all $h \in \mathcal{D}_*$

$$(3.4) \quad \|(J^{**}J^*)^{1/2}h\|^2 = \sup \{|(h, T^*Tf)|^2 : f \in \text{dom } T^*T, \|Tf\|^2 \leq 1\}.$$

Proof. First we describe the domain of J^* according to the definition of J as follows:

$$\begin{aligned} \text{dom } J^* &= \{h \in \mathfrak{H} : |(J(Sf), h)|^2 \leq m_h \cdot \langle Sf, Sf \rangle \text{ for all } f \in \text{dom } S\} \\ &= \{h \in \mathfrak{H} : |(T^*Tf, h)|^2 \leq m_h \cdot \|Tf\|^2 \text{ for all } f \in \text{dom } T^*T\}. \end{aligned}$$

It is well known that for any densely defined closed operator A between Hilbert spaces equality $\text{dom}(A^*A)^{1/2} = \text{dom } A$ holds. Thus the first part of our lemma follows. On the other hand $\text{ran } S$ is dense in \mathfrak{K}_T so that for all $h \in \mathfrak{K}_*$

$$\begin{aligned} \|(J^{**}J^*)^{1/2}h\|^2 &= \langle J^*h, J^*h \rangle \\ &= \sup \{ |\langle J^*h, Sf \rangle|^2 : f \in \text{dom } S, \langle Sf, Sf \rangle \leq 1 \} \\ &= \sup \{ |(h, T^*Tf)|^2 : f \in \text{dom } T^*T, \|Tf\|^2 \leq 1 \} \end{aligned}$$

and this proves formula (3.4). \square

Theorem 3.3. *Let T be a densely defined operator between Hilbert spaces \mathfrak{H} and \mathfrak{K} . Then $J^{**}J^*$ is the Krein-von Neumann extension of T^*T , the smallest among all positive selfadjoint extensions of T^*T .*

Proof. Assume that A is a positive selfadjoint extension of T^*T . We have to prove that $J^{**}J^* \leq A$. If $h \in \text{dom } A^{1/2}$ then for all $f \in \text{dom } T^*T$ we have that

$$\begin{aligned} |(T^*Tf, h)|^2 &= |(Af, h)|^2 \leq \|A^{1/2}h\|^2 \|A^{1/2}f\|^2 = \|A^{1/2}h\|^2 \|(T^*Tf, f)\| \\ &= \|A^{1/2}h\|^2 \|Tf\|^2. \end{aligned}$$

Hence $\text{dom } A^{1/2} \subseteq \mathcal{D}_*$ follows.

On the other hand, by applying Lemma 3.2 we obtain for each $h \in \text{dom } A^{1/2}$

$$\begin{aligned} \|A^{1/2}h\|^2 &= \sup \{ |(A^{1/2}h, A^{1/2}f)|^2 : f \in \text{dom } A^{1/2}, \|A^{1/2}f\|^2 \leq 1 \} \\ &\geq \sup \{ |(A^{1/2}h, A^{1/2}f)|^2 : f \in \text{dom } T^*T, \|A^{1/2}f\|^2 \leq 1 \} \\ &= \sup \{ |(h, Af)|^2 : f \in \text{dom } T^*T, (Af, f) \leq 1 \} \\ &= \sup \{ |(h, T^*Tf)|^2 : f \in \text{dom } T^*T, \|Tf\|^2 \leq 1 \} \\ &= \|(J^{**}J^*)^{1/2}h\|^2 \end{aligned}$$

and hence that $J^{**}J^* \leq A$. The proof is complete. \square

Observe that the Krein-von Neumann extension $J^{**}J^*$ of T^*T can be bounded although T is not continuous. For example, if T is a densely defined maximally singular operator, i.e. $\text{dom } T^* = \{0\}$ (see [31] and [38]), then T^*T coincides with the restriction of the zero operator to $\ker T$. So one obtains that $J^{**}J^* = 0$. On the other hand, there exist (not densely defined) bounded operators which do not have any bounded positive (self-adjoint) extension to \mathfrak{H} , moreover, which do not have any positive selfadjoint extension, see Example 2.3 of [50]. Nevertheless we notice that a bounded (not necessarily densely

defined) symmetric operator admits a norm-preserving selfadjoint extension to \mathfrak{H} , cf. [32] and [46].

The following theorem characterizes those operators for which the Krein–von Neumann extension of T^*T is bounded, c.f. also [45] and [46].

Theorem 3.4. *Let T be a densely defined operator between two Hilbert spaces \mathfrak{H} and \mathfrak{K} . Then the following statements are equivalent:*

- (a) *The Krein–von Neumann extension $J^{**}J^*$ of T^*T is bounded;*
- (b) *T^*T has a bounded positive (selfadjoint) extension;*
- (c) *There exists a nonnegative constant $m \geq 0$ such that*

$$\|T^*Tf\|^2 \leq m \cdot \|Tf\|^2$$

*for all $f \in \text{dom } T^*T$;*

- (d) $\mathcal{D}_* = \mathfrak{H}$.

Proof. It is clear that (a) implies (b). Assuming (b) there exists a bounded positive operator A extending T^*T . The Schwarz-inequality for bounded positive operators then implies that

$$\|T^*Th\|^2 = \|Ah\|^2 \leq \|A\| \cdot (Ah, h) = \|A\| \cdot (T^*Th, h) = \|A\| \cdot \|Th\|^2$$

for all $h \in \text{dom } T^*T$ which proves (c) by taking $m := \|A\|$. If we assume (c) and fix any $h \in \mathfrak{H}$ then for all $f \in \text{dom } T^*T$

$$|(T^*Tf, h)|^2 \leq \|h\|^2 \|T^*Tf\|^2 \leq m \cdot \|h\|^2 \|Tf\|^2.$$

Thus by choosing $m_h := m \cdot \|h\|^2$ one obtains that $h \in \mathcal{D}_*$ according to the definition of \mathcal{D}_* in (3.3). Finally, identity $\mathcal{D}_* = \text{dom } J^*$ and the Banach closed graph theorem show that assumption (d) implies that J^* is continuous together with $J^{**}J^*$. The proof is complete. \square

Corollary 3.5. *Let T be densely defined and closable operator between two Hilbert spaces \mathfrak{H} and \mathfrak{K} . The following statements are equivalent*

- (a) *T is bounded;*
- (b) *T^*T^{**} is bounded;*
- (c) *There is a constant $m \geq 0$ such that*

$$\|T^*T^{**}h\|^2 \leq m \cdot \|T^{**}h\|^2$$

*for all $h \in \text{dom } T^*T^{**}$.*

Proof. It is obvious that (a) implies (b). If (b) is assumed then identity

$$\text{dom } T^{**} = \text{dom}(T^*T^{**})^{1/2}$$

yields that T^{**} is bounded as well as T^* and thus (c) holds by taking $m = \|T^*\|^2$. Finally, since T is closable, T^*T^{**} is selfadjoint so that it coincides with its Krein–von Neumann extension $J^{**}J^*$ that is bounded by Theorem 3.4 in the case when (c) is assumed. This implies among others that T^{**} is defined on the entire Hilbert space \mathfrak{H} and thus it is continuous by the Banach closed graph theorem. \square

3.2. The Friedrichs extension of T^*T

In this section we consider the case when the Friedrichs extension $(T^*T)_F$, the largest among all positive selfadjoint extensions of T^*T exists. It is easy to see that then $\text{dom } T^*T$ has to be dense in \mathfrak{H} . On the other hand the density of T^*T is also sufficient as it was proved by several authors, c.f. for instance [39], [40] or [48]. We will not use these results.

If T^*T is assumed to be densely defined then the restriction Q of J^* to $\text{dom } T^*T$ is also densely defined and closable such that its closure Q^{**} has also the property in (3.2):

$$(3.5) \quad Q^{**}h = Sh \quad (h \in \text{dom } T^*T).$$

Consequently, Q^*Q^{**} is positive selfadjoint extension of T^*T . In Theorem 3.7 it will be shown that Q^*Q^{**} equals the Friedrichs extension of T^*T . But first we need the following lemma that describes the domain of the square root of Q^*Q^{**} .

Lemma 3.6. *Let T be densely defined operator between Hilbert spaces \mathfrak{H} and \mathfrak{K} such that $\text{dom } T^*T$ is dense in \mathfrak{H} . Then $\text{dom } (Q^*Q^{**})^{1/2} = \mathcal{D}_{**}$ where*

$$(3.6) \quad \mathcal{D}_{**} := \{h \in \mathfrak{H} : \exists \{f_n\}_{n=1}^\infty \subseteq \text{dom } T^*T, f_n \rightarrow h, \{Tf_n\}_{n=1}^\infty \text{ converges}\}.$$

Proof. Using the characteristic property (3.5) of Q^{**} and the well known fact that Q^{**} is the closure of Q we obtain the following identities

$$\begin{aligned} \text{dom } Q^{**} &= \{h \in \mathfrak{H} : \exists \{f_n\}_{n=1}^\infty \subseteq \text{dom } Q, f_n \rightarrow h, \{Qf_n\}_{n=1}^\infty \subseteq \mathfrak{K}_r \text{ converges}\} \\ &= \{h \in \mathfrak{H} : \exists \{f_n\}_{n=1}^\infty \subseteq \text{dom } T^*T, f_n \rightarrow h, \{Tf_n\}_{n=1}^\infty \text{ converges}\} \\ &=: \mathcal{D}_{**}. \end{aligned}$$

The proof follows then according to identity $\text{dom } (Q^*Q^{**})^{1/2} = \text{dom } Q^{**}$. \square

Theorem 3.7. *Let T be densely defined operator between Hilbert spaces \mathfrak{H} and \mathfrak{K} such that $\text{dom } T^*T$ is dense. Then Q^*Q^{**} is the Friedrichs extension of T^*T , i.e. the largest among all positive selfadjoint extensions of T^*T .*

Proof. We have already seen that Q^*Q^{**} is a positive selfadjoint extension of T^*T . Let A be any positive selfadjoint extension of T^*T and let $h \in \mathcal{D}_{**}$. Then there is a sequence $\{f_n\}_{n=1}^\infty$ in $\text{dom } T^*T$ such that $f_n \rightarrow h$ and $\{Tf_n\}_{n=1}^\infty$ is Cauchy-sequence in \mathfrak{K} . Since the

following equalities

$$\|Tf\|^2 = (T^*Tf, f) = (Af, f) = \|A^{1/2}f\|^2$$

hold for all $f \in \text{dom } T^*T$ we conclude that $\{A^{1/2}f_n\}_{n=1}^\infty$ is also Cauchy-sequence in \mathfrak{H} and thus $h \in \text{dom } A^{1/2}$ according to the closedness of $A^{1/2}$. On the other hand

$$\begin{aligned} \|(Q^*Q^{**})^{1/2}h\|^2 &= \langle Q^{**}h, Q^{**}h \rangle = \lim_{n \rightarrow \infty} \|Tf_n\|^2 = \lim_{n \rightarrow \infty} \|A^{1/2}f_n\|^2 \\ &= \|A^{1/2}h\|^2 \end{aligned}$$

which implies $A \leq Q^*Q^{**}$. The proof is complete. \square

Assume that T^*T is densely defined and that A is a positive selfadjoint extension of T^*T . According to the proof of Theorem 3.7 we can conclude that $\mathcal{D}_{**} \subseteq \text{dom } A^{1/2}$ and that

$$\|A^{1/2}h\|^2 = \langle Q^{**}h, Q^{**}h \rangle$$

holds for each $h \in \mathcal{D}_{**}$. This yields the following corollary:

Corollary 3.8. *Assume that T is a densely defined operator between Hilbert spaces such that $\text{dom } T^*T$ is dense. For a positive selfadjoint extension A of T^*T the following statements are equivalent:*

- (a) A equals the Friedrichs extension of T^*T ;
- (b) $\text{dom } A^{1/2} \subseteq \mathcal{D}_{**}$.

Theorem 3.9. *Let T be a densely defined operator between two Hilbert spaces \mathfrak{H} and \mathfrak{K} . Let S denote the restriction of T to $\text{dom } T^*T$. Then the following two assertions hold:*

- (a) S is closable.
- (b) *If $\text{dom } T^*T$ is assumed to be dense in \mathfrak{H} then the Friedrichs extension Q^*Q^{**} of T^*T equals S^*S^{**} .*

Proof. Let $\{f_n\}_{n=1}^\infty$ be a sequence from $\text{dom } S$ that tends to zero such that $\{Sf_n\}_{n=1}^\infty$ converges to a vector $k \in \mathfrak{K}$. According to Theorem 3.1, T^*T admits its Krein-von Neumann extension $J^{**}J^*$. From identity (3.4) we obtain that

$$\|(J^{**}J^*)^{1/2}f\|^2 = \|Sf\|^2$$

holds for each $f \in \text{dom } T^*T$. This equality shows that $\{(J^{**}J^*)^{1/2}f_n\}_{n=1}^\infty$ is a Cauchy-sequence whose limit point equals zero according to the closedness of $(J^{**}J^*)^{1/2}$. Thus

$$\|k\| = \lim_{n \rightarrow \infty} \|Sf_n\| = \lim_{n \rightarrow \infty} \|(J^{**}J^*)^{1/2}f_n\| = 0$$

which proves that S is closable.

Assume now that $\text{dom } T^*T$ is dense in \mathfrak{H} and that A is a positive selfadjoint extension of T^*T . Due to Corollary 3.8 (b) will be obtained by showing that $\text{dom } S^{**} \subseteq \mathcal{D}_{**}$. So let us suppose that $h \in \text{dom } S^{**}$. Then there is a sequence $\{f_n\}_{n=1}^\infty$ in $\text{dom } T^*T$ such that

$f_n \rightarrow h$ and $\{Sf_n\}_{n=1}^\infty$ converges. Since S is a restriction of T we obtain that $h \in \mathcal{D}_{**}$ which completes the proof. \square

We notice that an alternative proof for part (a) of Theorem 3.9 can be given which does not use Theorem 3.1 as follows. Let $\{f_n\}_{n=1}^\infty$ be a sequence from $\text{dom } S$ converging to zero so that $Sf_n \rightarrow k$ for some $k \in \mathfrak{K}$. Then for each $g \in \text{dom } T^*$ we find that

$$(g, k) = \lim_{n \rightarrow \infty} (g, Tf_n) = \lim_{n \rightarrow \infty} (T^*g, f_n) = 0.$$

In other words, $k \in (\text{dom } T^*)^\perp$ as well $Sf_n \in \text{dom } T^*$ for all $n \in \mathbb{N}$ so that

$$(k, k) = \left(\lim_{n \rightarrow \infty} Tf_n, k \right) = \lim_{n \rightarrow \infty} (Tf_n, k) = 0.$$

Hence S is closable.

3.3. The selfadjointness of T^*T

The von Neumann theorem, among others, states that T^*T is selfadjoint operator whenever T is closed and densely defined operator between Hilbert spaces. Obviously, the densely definedness of T in von Neumann's theorem is essential. But T^*T can be selfadjoint without assuming T to be closed: If T is densely defined and closed then the restriction S of T to $\text{dom } T^*T$ is also densely defined so that its closure S^{**} equals T . Consequently,

$$S^*S = T^*T$$

and hence S^*S is selfadjoint. Nevertheless, equality $S = S^{**} = T$ can be happen just whenever T is continuous, see [51] Lemma 2.1. Thus S is never closed (but closable) when T is unbounded. On the other hand it is not sufficient to assume T to be closable. Moreover, in Theorem 3.14 it will be shown that any densely defined positive operator can be given in the form T^*T where T is densely defined and closable.

In this section we give necessary and sufficient conditions for the selfadjointness of T^*T . This can happen only when T^*T coincides with its Krein–von Neumann extension $J^{**}J^*$, i.e. only when they have common domain of definition. On the other hand, thanks to Arens [4] equivalently whenever the operators in question have common ranges and kernels.

Lemma 3.10. *Let T be a densely defined operator between Hilbert spaces \mathfrak{H} and \mathfrak{K} . Then $\text{ran}(J^{**}J^*)^{1/2} = \text{ran } J^{**} = \mathcal{R}_*$ where*

$$(3.7) \quad \mathcal{R}_* := \{h \in \mathfrak{H} : \exists \{f_n\}_{n=1}^\infty \subseteq \text{dom } T^*T, \{Tf_n\}_{n=1}^\infty \text{ converges, } T^*Tf_n \rightarrow h\}$$

Proof. The first equality of the statement is well known. To prove $\text{ran } J^{**} = \mathcal{R}_*$ observe that J^{**} is the closure of J and thus

$$\begin{aligned} \text{ran } J^{**} &= \{h \in \mathfrak{H} : \exists \{g_n\}_{n=1}^\infty \subseteq \text{dom } J, \{g_n\}_{n=1}^\infty \text{ converges, } Jg_n \rightarrow h\} \\ &= \{h \in \mathfrak{H} : \exists \{f_n\}_{n=1}^\infty \subseteq \text{dom } T^*T, \{Tf_n\}_{n=1}^\infty \text{ converges, } T^*Tf_n \rightarrow h\} \\ &=: \mathcal{R}_*. \end{aligned}$$

The proof is complete. \square

Theorem 3.11. *Let T be a densely defined operator between Hilbert spaces \mathfrak{H} and \mathfrak{K} . The operator T^*T in the Hilbert space \mathfrak{H} is selfadjoint if and only if one of the following conditions holds:*

$$(3.8) \quad \text{dom } T^*T = \{g \in \mathcal{D}_* : \exists h \in \mathcal{R}_*, (T^*Tf, g) = (f, h) \text{ for all } f \in \text{dom } T^*T\}.$$

$$(3.9) \quad \begin{cases} \ker T^*T = (\text{ran } T^*T)^\perp, \\ \text{ran } T^*T = \{h \in \text{ran } T_* : \sup\{|(f, h)| : f \in \text{dom } T^*T, \|T^*Tf\| \leq 1\} < \infty\}, \end{cases}$$

where T_* stands for the closure of the restriction of T^* to $\text{dom } T^* \cap \text{ran } T$.

Proof. We know from Theorem 3.1 that $J^{**}J^*$ is a positive selfadjoint extension of T^*T . Thus, as it was mentioned in the introduction of this section, T^*T is selfadjoint if and only of $\text{dom } T^*T = \text{dom } J^{**}J^*$. According again to the fact that J^{**} equals the closure of J we obtain the following line of identities:

$$\begin{aligned} \text{dom } J^{**}J^* &= \{g \in \text{dom } J^* : J^*g \in \text{dom } J^{**}\} \\ &= \{g \in \mathcal{D}_* : \exists h \in \mathfrak{H}, \exists \{f_n\}_{n=1}^\infty \subseteq \text{dom } T^*T, Tf_n \rightarrow J^*g, J(Tf) \rightarrow h\} \\ &= \{g \in \mathcal{D}_* : \exists h \in \mathcal{R}_*, \exists \{f_n\}_{n=1}^\infty \subseteq \text{dom } T^*T, Tf_n \rightarrow J^*g, T^*Tf \rightarrow h\} \\ &= \{g \in \mathcal{D}_* : \exists h \in \mathcal{R}_*, (T^*Tf, g) = (f, h) \text{ for all } f \in \text{dom } T^*T\}. \end{aligned}$$

The last equality above follows from Lemma 3.10 and from identities

$$\begin{aligned} (T^*Tf, g) &= (J(Tf), g) = \langle Tf, J^*g \rangle = \left\langle Tf, \lim_{n \rightarrow \infty} Tf_n \right\rangle = \lim_{n \rightarrow \infty} (f, T^*Tf_n) \\ &= \left(f, \lim_{n \rightarrow \infty} T^*Tf_n \right) = (f, h). \end{aligned}$$

Thus the selfadjointness of T^*T is equivalent to condition (3.8).

Observe that Lemma 3.10 yields in particular that $\overline{\text{ran } J^{**}} = \overline{\text{ran } T^*T}$. Thus

$$(3.10) \quad \ker J^{**}J^* = \ker J^* = (\text{ran } J^{**})^\perp = (\text{ran } T^*T)^\perp.$$

On the other hand, following the argument of the proof of Theorem 2 in [48] we obtain that

$$\begin{aligned}
\text{ran } J^{**}J^* &= \{J^{**}g : g \in \text{dom } J^{**} \cap \text{ran } J^*\} \\
&= \{J^{**}g : g \in \text{ran } J^*, \exists \{f_n\}_{n=1}^\infty \subseteq \text{dom } T^*T, Tf_n \rightarrow g, J(Tf_n) \rightarrow J^{**}g\} \\
&= \{h \in \mathfrak{H} : \exists g \in \text{ran } J^*, \exists \{f_n\}_{n=1}^\infty \subseteq \text{dom } T^*T, Tf_n \rightarrow g, T^*Tf_n \rightarrow h\} \\
&= \{T^*g : g \in \text{ran } J^* \cap \text{dom } (\overline{T^*|_{\text{dom } T^* \cap \text{ran } T}})\} \\
&= \{T^*g : g \in \text{ran } J^* \cap \text{dom } T_*\}.
\end{aligned}$$

The range of J^* can be described by using Theorem 1 of [44] as follows:

$$\begin{aligned}
\text{ran } J^* &= \{g \in \mathfrak{K}_T : \sup\{|\langle Tf, g \rangle| : f \in \text{dom } T^*T, \|J(Tf)\|^2 \leq 1\} < \infty\} \\
&= \{g \in \overline{\text{ran } S} : \sup\{|\langle Tf, g \rangle| : f \in \text{dom } T^*T, \|T^*Tf\|^2 \leq 1\} < \infty\}.
\end{aligned}$$

Since $\text{dom } T^* \cap \text{ran } T = \text{ran } S$ we conclude that $\text{dom } T_* \subseteq \overline{\text{ran } S}$ and so

$$\begin{aligned}
\text{ran } J^{**}J^* &= \{T^*g : g \in \text{dom } T_*, \sup\{|\langle Tf, g \rangle| : f \in \text{dom } T^*T, \|T^*Tf\| \leq 1\} < \infty\} \\
&= \{h \in \text{ran } T_* : \sup\{|\langle f, h \rangle| : f \in \text{dom } T^*T, \|T^*Tf\| \leq 1\} < \infty\}.
\end{aligned}$$

This identity above together with (3.10) show that properties (3.9) are satisfied if and only if T^*T is selfadjoint. \square

3.4. Some consequences of the construction

If T is a densely defined closable operator between two Hilbert spaces then, thanks to von Neumann's theorem T^*T^{**} is the most natural positive selfadjoint extension of T^*T . Below we give on one hand a sufficient condition for closability of densely defined operators, and on the other hand it will be shown that T^*T^{**} can be constructed along the procedure above for the Krein–von Neumann and the Friedrichs extensions of T^*T .

Theorem 3.12. *Let T be a densely defined operator between two Hilbert spaces \mathfrak{H} and \mathfrak{K} such that*

$$(3.11) \quad \text{ran } T \subset \overline{\text{ran}(T|_{\text{dom } T^*T})}.$$

*Then T is closable and $T^*T^{**} = V^*V^{**}$ where V stands for the restriction of J^* to $\text{dom } T$.*

Proof. Condition (3.11) can also be formulated as follows

$$\|Th\|^2 = \sup\{|\langle Th, Tf \rangle|^2 : f \in \text{dom } T^*T, \|Tf\| \leq 1\}$$

for all $h \in \text{dom } T$. Since $\text{dom } T \subset \mathscr{D}_*$, equality (3.4) implies that

$$\|Th\|^2 = \|(J^{**}J^*)^{1/2}h\|^2 = \langle Vh, Vh \rangle$$

for all $h \in \text{dom } T$. Since V is closable and has the same domain as T it follows that T is closable too, $\text{dom } T^{**} = \text{dom } V^{**}$ and

$$\|(T^*T^{**})^{1/2}h\|^2 = \|T^{**}h\|^2 = \langle V^{**}h, V^{**}h \rangle = \|(V^*V^{**})^{1/2}h\|^2$$

for all $h \in \text{dom}(T^*T^{**})^{1/2} = \text{dom}(V^*V^{**})^{1/2}$. Thus $V^*V^{**} = T^*T^{**}$. \square

It is well known that for any densely defined closed operator from \mathfrak{H} into \mathfrak{K} the following identity holds:

$$(3.12) \quad \text{graph } T = \overline{\text{graph}(T|_{\text{dom } T^*T})}.$$

The closure is taken in the Cartesian product $\mathfrak{H} \times \mathfrak{K}$. Property (3.12) is also expressed by saying "dom T^*T is core for T ". The next corollary gives in particular some conversed version of this statement.

Corollary 3.13. *Let T be a densely defined operator between two Hilbert spaces such that*

$$(3.13) \quad \overline{\text{graph } T} = \overline{\text{graph}(T|_{\text{dom } T^*T})}.$$

*Then T is closable and the Friedrichs extension of T^*T equals T^*T^{**} .*

Proof. According to Theorem 3.9 the right side of (3.13) is the graph of an operator, thus T is closable indeed. On the other hand, it is easy to check that identity (3.13) implies among others that $\text{dom } T^*T$ is dense and that condition (3.11) of Theorem 3.12 holds. Hence, using the results and notations of Theorem 3.9, the Friedrichs extension Q^*Q^{**} of T^*T exists and

$$Q^*Q^{**} = S^*S^{**} = T^*T^{**}.$$

The proof is complete. \square

We close this chapter with the following theorem describing all densely defined positive operators and their Friedrichs extension:

Theorem 3.14. *If A is a densely defined positive operator in a Hilbert space \mathfrak{H} then there is densely defined closable operator T acting in \mathfrak{H} such that*

$$A = T^*T$$

*and that T^*T^{**} is the Friedrichs extension of A .*

Proof. Let \mathfrak{H}_A denote the Hilbert completion of $\text{ran } A$ endowed with the inner product

$$\langle Af, Ag \rangle_A := \langle Af, g \rangle \quad (f, g \in \text{dom } A).$$

The canonical embedding operator \mathfrak{J} of $\text{ran } A \subseteq \mathfrak{H}_A$ into \mathfrak{H} defined by the relation

$$Af \mapsto Af \quad (f \in \text{dom } A)$$

is then a densely defined operator with the domain $\text{ran } A$. By applying the Cauchy-Schwarz inequality to the inner product $\langle \cdot, \cdot \rangle_A$ one obtains that

$$|(Af, g)|^2 \leq (Af, f) \cdot (Ag, g) = \langle Af, Af \rangle_A \cdot \langle Ag, g \rangle \quad (f, g \in \text{dom } A)$$

and thus that the linear functional

$$Af \mapsto (\mathfrak{J}(Af), g) = (Af, g)$$

is continuous on $\text{ran } A \subseteq \mathfrak{H}_A$ for all $g \in \text{dom } A$ with norm bound $(Ag, g)^{1/2}$. This yields $\text{dom } A \subseteq \text{dom } \mathfrak{J}^*$ and that \mathfrak{J} is closable in particular. Let R denote the restriction of \mathfrak{J}^* to $\text{dom } A$. Then R is densely defined and closable so that for all $f, g \in \text{dom } R$

$$\langle Rf - Af, Ag \rangle_A = \langle \mathfrak{J}^*f - Af, Ag \rangle_A = (f, Ag) - \langle Af, Ag \rangle_A = 0$$

holds. That yields

$$(3.14) \quad Rf = Af \in \mathfrak{H}_A \quad (f \in \text{dom } R)$$

taking into account that $\text{ran } A$ forms a dense linear manifold in \mathfrak{H}_A by definition. Since R is a restriction of \mathfrak{J}^* , R^* extends \mathfrak{J}^{**} . According to property (3.14) of R we conclude that

$$\text{ran } R \subseteq \text{dom } \mathfrak{J} \subseteq \text{dom } R^*.$$

Thus $\text{dom } R^*R = \text{dom } A$ and according again to (3.14)

$$R^*Rf = R^*(Af) = \mathfrak{J}(Af) = Af$$

for all $f \in \text{dom } A$. Consequently, $R^*R = A$.

The following relation

$$Af \mapsto (R^*R^{**})^{1/2}f \quad (f \in \text{dom } A),$$

denoted by U , defines an isometry from \mathfrak{H}_A into \mathfrak{H} with dense domain $\text{dom } U = \text{ran } A$:

$$\|U(Af)\|^2 = (R^*R^{**}f, f) = (R^*Rf, f) = (Af, f) = \langle Af, Af \rangle_A.$$

Thus its unique continuous extension U^{**} to \mathfrak{H} is also an isometry so that

$$U^*U^{**} = I_A,$$

where I_A stands for the identity operator of \mathfrak{H}_A . Finally, $T := U^{**}R$ is a densely defined operator acting in \mathfrak{H} with the domain $\text{dom } T = \text{dom } A$ and having the adjoint $T^* = R^*U^*$ so that

$$T^*T = R^*U^*U^{**}R = R^*R = A.$$

Since R is closable and U^{**} is an isometry, it follows that

$$\langle Rf, Rf \rangle_A = \|Tf\|^2$$

for all $f \in \text{dom } R = \text{dom } T$. Hence T is closable and its closure T^{**} satisfies

$$(3.15) \quad \text{dom } T^{**} = \text{dom } R^{**}.$$

It only remains to show that T^*T^{**} is the Friedrichs extension of $A = T^*T$. It is clear that T^*T^{**} is a positive selfadjoint extension of A . Hence, according to Corollary 3.8 and identity (3.15), it is enough to show that

$$\text{dom } R^{**} = \mathcal{D}_{**}.$$

Since R^{**} is the closure of R , it follows that

$$\begin{aligned} \text{dom } R^{**} &= \{h \in \mathfrak{H} : \exists \{f_n\}_{n=1}^\infty \subseteq \text{dom } R, f_n \rightarrow h, \langle R(f_n - f_m), R(f_n - f_m) \rangle_A \rightarrow 0\} \\ &= \{h \in \mathfrak{H} : \exists \{f_n\}_{n=1}^\infty \subseteq \text{dom } A, f_n \rightarrow h, (A(f_n - f_m), f_n - f_m) \rightarrow 0\} \\ &= \{h \in \mathfrak{H} : \exists \{f_n\}_{n=1}^\infty \subseteq \text{dom } A, f_n \rightarrow h, (T(f_n - f_m), T(f_n - f_m)) \rightarrow 0\} \\ &= \{h \in \mathfrak{H} : \exists \{f_n\}_{n=1}^\infty \subseteq \text{dom } T^*T, f_n \rightarrow h, \{Tf_n\}_{n=1}^\infty \text{ converges}\} \\ &=: \mathcal{D}_{**}. \end{aligned}$$

The proof is complete. □

CHAPTER 4

Operator extensions with closed range

4.1. Introduction

Throughout this chapter \mathfrak{H} is a complex Hilbert space with the inner product (\cdot, \cdot) and A is a (not necessarily densely defined or closed) operator on it which is assumed to be positive in the usual sense that

$$(Ah, h) \geq 0 \quad (h \in \operatorname{dom} A).$$

According to Theorem 1 of Sebestyén and Stochel [49], A admits a positive selfadjoint extension if and only if the linear manifold

$$(4.1) \quad \mathcal{D}_*(A) := \{k \in \mathfrak{H} : |(Ah, k)|^2 \leq m_k \cdot (Ah, h) \text{ for all } h \in \operatorname{dom} A\}$$

is dense in \mathfrak{H} . If this is the case then the Krein–von Neumann extension A_N , the smallest among the set of all positive selfadjoint extensions of A also exists. We briefly recall the method of [49] for giving the Krein–von Neumann extension. Equip the range space $\operatorname{ran} A$ with the semi-inner product $\langle \cdot, \cdot \rangle_A$ defined as follows

$$\langle Ah, Ak \rangle_A = (Ah, k) \quad (h, k \in \operatorname{dom} A).$$

In fact, according to the density of $\mathcal{D}_*(A)$, it turns out that $\langle \cdot, \cdot \rangle_A$ is an inner product on $\operatorname{ran} A$. Let \mathfrak{H}_A denote the completion of this pre-Hilbert space so that $\operatorname{ran} A$ forms a dense linear manifold in \mathfrak{H}_A by definition. The natural embedding operator J_A of $\operatorname{ran} A \subseteq \mathfrak{H}_A$ into \mathfrak{H} defined by the identification

$$(4.2) \quad Ah \mapsto Ah \quad (h \in \operatorname{dom} A)$$

is then a densely defined operator between the Hilbert spaces \mathfrak{H}_A and \mathfrak{H} so that its adjoint J_A^* has the dense domain $\operatorname{dom} J_A^* = \mathcal{D}_*(A)$. According to the Schwarz-inequality one obtains easily that $\operatorname{dom} A$ is always contained by $\mathcal{D}_*(A)$ and that J_A^* acts between \mathfrak{H} and \mathfrak{H}_A with the characteristic extension property

$$(4.3) \quad J_A^* h = Ah \in \mathfrak{H}_A \quad (h \in \operatorname{dom} A).$$

Since $\mathcal{D}_*(A)$ is dense by assumption we obtain that J_A is closable, furthermore $J_A^{**} J_A^*$ is positive selfadjoint end extends A as follows:

$$J_A^{**} J_A^* h = J_A^{**} (Ah) = J_A(Ah) = Ah \quad (h \in \operatorname{dom} A).$$

It turns also out that $J_A^{**} J_A^*$ coincides with the Krein–von Neumann extension of A , cf. [48] or [49].

In the present chapter we characterize those positive operators which have any positive selfadjoint extensions with closed range. It turns out that this can only be happen when the Krein-von Neumann extension of A has closed range. We also provide the Moore-Penrose pseudoinverse of A_N in terms of factorization used above for giving the Krein-von Neumann extension of A ; see also [60] for the case when A_N is bounded.

In the last section another special class of positive selfadjoint extensions is studied from this point of view: the class of so called *extremal* extensions, cf. [5, 6] or [23] for case of nonnegative linear relations. In particular the Friedrichs extension of A is also considered when A is densely defined; we provide necessary and sufficient conditions on A which ensure that the Friedrichs extension of A has closed range. At the same time we prove that all of the extremal extensions of A have closed range whenever the Friedrichs extension has. As an application of our results we offer some new characterizations of essentially selfadjoint operators in terms of their Krein-von Neumann and Friedrichs extensions.

4.2. Operators with closed domain or range

In this section we give some characterizations for closed range operators. But first we need the following lemma characterizing those densely defined closed operators which are also bounded.

Lemma 4.1. *For a given densely defined closed operator T between the Hilbert spaces \mathfrak{H} and \mathfrak{K} the following statements are equivalent:*

- (i) T is bounded;
- (ii) $\operatorname{dom} TT^* = \operatorname{dom} T^*$;
- (i') T^* is bounded;
- (ii') $\operatorname{dom} T^*T = \operatorname{dom} T$.

Proof. It is well known that for a densely defined closed operator T the following identity

$$(4.4) \quad \operatorname{graph} T^* \oplus \{V(\operatorname{graph} T)\}^\perp = \mathfrak{K} \times \mathfrak{H}$$

on the graphs of T and T^* holds where $V : \mathfrak{H} \times \mathfrak{K} \rightarrow \mathfrak{K} \times \mathfrak{H}$ denotes the isomorphism defined by the relation

$$V\{h, k\} = \{-k, h\} \quad (\{h, k\} \in \mathfrak{H} \times \mathfrak{K}).$$

This implies among others that for each $\{k, h\} \in \mathfrak{K} \times \mathfrak{H}$ the system of equations

$$\begin{cases} k = y - Tx, \\ h = x + T^*y \end{cases}$$

has unique solution $\{x, y\}$, $x \in \operatorname{dom} T$, $y \in \operatorname{dom} T^*$. Consequently, it follows that

$$(4.5) \quad \mathfrak{H} = \operatorname{dom} T + \operatorname{ran} T^* \quad \text{and} \quad \mathfrak{K} = \operatorname{dom} T^* + \operatorname{ran} T.$$

Note that assumption (ii) means that $\text{ran } T^* \subseteq \text{dom } T$. Thus (4.5) and the Banach closed graph theorem imply boundedness on T . Similarly can be proved that (ii') implies (i'). Implications (i) \Rightarrow (ii), (i') \Rightarrow (ii') and the equivalence of (i) and (i') are clear. The proof is complete. \square

Corollary 4.2. *Let T be a densely defined closed operator in the Hilbert space \mathfrak{H} such that $\text{dom } T \subseteq \text{dom } T^*$. Then T is bounded if and only if $\text{dom } T = \text{dom } T^2$.*

Proof. From relationship $\text{dom } T = \text{dom } T^2$ we obtain that

$$\begin{aligned} \text{dom } T &= \text{dom } T^2 = \{h \in \text{dom } T : Th \in \text{dom } T\} \\ &\subseteq \{h \in \text{dom } T : Th \in \text{dom } T^*\} = \text{dom } T^*T, \end{aligned}$$

so Lemma 4.1 applies. \square

The next result characterizing continuous normal operators is taken from Dieudonné [13]:

Corollary 4.3. *A normal operator N in the Hilbert space \mathfrak{H} is bounded if and only if N and N^2 have the same domain.*

Proof. It is well known that for each densely defined closed operator T between two Hilbert spaces the equality $\text{dom}(T^*T)^{1/2} = \text{dom } T$ holds. This fact and identity $N^*N = NN^*$ imply that

$$\text{dom } N = \text{dom}(N^*N)^{1/2} = \text{dom}(NN^*)^{1/2} = \text{dom } N^*,$$

and so Corollary 4.2 applies. \square

Corollary 4.4. *A positive selfadjoint operator A in the Hilbert space \mathfrak{H} is bounded if and only if $\text{dom } A = \text{dom } A^{1/2}$.*

Proof. Apply Corollary 4.3 to the normal operator $A^{1/2}$. \square

The next theorem is an extension of Dixmier's result that a bounded positive operator A has closed range if and only if A and its square root $A^{1/2}$ have common ranges; see [14], [17] or [60] for bounded A . We notice that all of these results use the boundedness of A .

Theorem 4.5. *Let A be positive selfadjoint operator in the Hilbert space \mathfrak{H} . Then A has closed range if and only if $\text{ran } A = \text{ran } A^{1/2}$.*

Proof. Assume first that A has closed range. From the identity $\ker A = \ker A^{1/2}$ it follows that

$$\text{ran } A = \ker A^\perp = \overline{\text{ran } A^{1/2}} \supset \text{ran } A^{1/2} \supseteq \text{ran } A,$$

and hence $\text{ran } A = \text{ran } A^{1/2}$.

Conversely, assume that $\text{ran } A$ equals $\text{ran } A^{1/2}$. If \mathcal{R} denotes their common range, S and T stand for the restrictions of A and $A^{1/2}$ to \mathcal{R} , respectively, then one can prove that both of the restricted maps are injective positive selfadjoint operators in the Hilbert space $\overline{\mathcal{R}}$ with the common range space \mathcal{R} and satisfy the equality

$$(4.6) \quad S^{1/2} = T.$$

Consequently, their inverses have the common domain \mathcal{R} . Since both S^{-1} and T^{-1} are positive selfadjoint operators in $\overline{\mathcal{R}}$ so that

$$(T^{-1})^2 = S^{-1},$$

from the uniqueness of the positive selfadjoint square root it follows that

$$(S^{-1})^{1/2} = T^{-1}.$$

According to Corollary 4.4, S^{-1} is bounded and so its domain \mathcal{R} must be closed. The proof is complete. \square

We close this section with a result which will be useful in the later ones by deciding whether a given operator T has closed range. The equivalence of parts (i) and (iv) of this theorem is first proved by Banach [7] for bounded operators. For unbounded operators it is proved by several authors; see Browder [11], Joichi [27], Kato [30] and Rota [43].

Theorem 4.6. *For a given densely defined closed operator T between the Hilbert spaces \mathfrak{H} and \mathfrak{K} the following statements are equivalent:*

- (i) T has closed range;
- (ii) TT^* has closed range;
- (iii) T and TT^* have equal ranges;
- (iv) T^* has closed range;
- (v) T^*T has closed range;
- (vi) T^* and T^*T have equal ranges.

Proof. Assume first (i). From identity $\text{ran } T = \text{ran}(TT^*)^{1/2}$ and by applying Theorem 4.5 to $A = (TT^*)^{1/2}$ it follows that

$$\text{ran}(TT^*)^{1/2} = \text{ran}(TT^*)^{1/4}.$$

Assertions (ii) and (iii) will be proved by showing equality $\text{ran } A = \text{ran } A^2$. Since $\text{ran } A = \text{ran } A^{1/2}$, due to Theorem 2 of [44] there exists a linear operator S in \mathfrak{K} with domain $\text{dom } A^{1/2}$ and range $\text{ran } S \subseteq \text{dom } A$ such that

$$A^{1/2}h = ASh \quad (h \in \text{dom } A^{1/2}).$$

Therefore we obtain that for all $h \in \text{dom } A$

$$Ah = A^{1/2}(ASh) = A^{3/2}Sh,$$

and hence that $\text{ran } A \subseteq \text{ran } A^{3/2}$. Similarly, for all $h \in \text{dom } A^{3/2}$ we have that

$$A^{3/2}h = A(ASH) = A^2Sh,$$

and hence the following line of range inclusions holds:

$$\text{ran } A \subseteq \text{ran } A^{3/2} \subseteq \text{ran } A^2.$$

Note that both of conditions (ii) and (iii) follow the range equality

$$(4.7) \quad \text{ran } TT^* = \text{ran}(TT^*)^{1/2} = \text{ran } T,$$

and hence according to Theorem 4.5 either of them imply (i). Replacing T by T^* above, the equivalence of statements (iv), (v) and (vi) is clear. Assume now (i) and prove (vi). Since $\text{ran } T$ is closed, the following orthogonal decomposition of the Hilbert space \mathfrak{H} holds:

$$\mathfrak{H} = \text{ran } T \oplus \ker T^*.$$

This means among others that for each k from $\text{dom } T^*$ there exists a pair of vectors $h \in \text{dom } T$ and $k_1 \in \ker T^*$ with $k = Th + k_1$. Therefore we have that

$$T^*k = T^*(Th + k_1) = T^*Th,$$

and hence the range equality $\text{ran } T^* = \text{ran } T^*T$ holds. A similar reasoning shows that (iv) implies (iii). The proof is complete. \square

4.3. Krein–von Neumann extension with closed range

In the first theorem of this section we characterize those positive operators which have any positive selfadjoint extensions with closed range.

Theorem 4.7. *Let A be a (not necessarily densely defined or closed) positive operator in the Hilbert space \mathfrak{H} such that the linear manifold $\mathcal{D}_*(A)$ defined in (4.1) is dense, i.e. the Krein–von Neumann extension of A exists. The following statements are equivalent:*

- (i) *There exists a constant $m_A \geq 0$ such that for each $h \in \text{dom } A$ the following inequality holds:*

$$(4.8) \quad (Ah, h) \leq m_A \cdot \|Ah\|^2.$$

- (ii) *The Krein–von Neumann extension of A has closed range.*
- (iii) *There is a positive selfadjoint extension \tilde{A} of A that has closed range.*

Proof. In the introduction of this chapter we saw that the density of $\mathcal{D}_*(A)$ guarantees the existence of the Krein–von Neumann extension A_N of A . At first we prove the equivalence of (i) and (ii). If we assume that A_N has closed range and J_A denotes the map defined in (5.3) then we obtain from identity $J_A^{**}J_A^* = A_N$ and Theorem 4.6 that the range of J_A^*

is also closed in the auxiliary Hilbert space \mathfrak{H}_A . On the other hand, according to formula (5.4) we have the following range inclusion

$$\text{ran } A \subseteq \text{ran } J_A^*,$$

where $\text{ran } A$ forms a dense linear manifold in \mathfrak{H}_A . Consequently, due to Theorem 4.6 the positive selfadjoint operator $\widehat{A} := J_A^* J_A^{**}$ acting in \mathfrak{H}_A is surjective and has then bounded inverse. This implies among others that its square root $\widehat{A}^{1/2}$ is bounded from below in the sense that

$$(4.9) \quad \langle \widehat{A}^{1/2} \xi, \widehat{A}^{1/2} \xi \rangle_A \geq \gamma \cdot \langle \xi, \xi \rangle_A \quad (\xi \in \text{dom } J_A^{**})$$

holds with some constant $\gamma > 0$. In particular, for $h \in \text{dom } A$

$$\begin{aligned} (Ah, Ah) &= (J_A^{**}(Ah), J_A^{**}(Ah)) \\ &= \langle \widehat{A}^{1/2}(Ah), \widehat{A}^{1/2}(Ah) \rangle_A \geq \gamma \cdot \langle Ah, Ah \rangle_A = \gamma \cdot (Ah, h), \end{aligned}$$

which yields inequality (4.8) by taking $m_A = 1/\gamma$.

Conversely, assume that inequality (4.8) is valid. Using the definition of J_A in (5.3) and that J_A^{**} is its closure, from inequality (4.8) we obtain that

$$\langle \widehat{A}^{1/2} \xi, \widehat{A}^{1/2} \xi \rangle_A = (J_A^{**} \xi, J_A^{**} \xi) \geq m_A^{-1} \cdot \langle \xi, \xi \rangle_A \quad (\xi \in \text{dom } J_A^{**}).$$

This means among others that \widehat{A} has bounded inverse on \mathfrak{H}_A and hence that

$$\text{ran } \widehat{A} = \text{ran } J_A^* J_A^{**} = \mathfrak{H}_A.$$

According to Theorem 4.6, the Krein–von Neumann extension $A_N = J_A^{**} J_A^*$ of A has closed range. Finally, assume that \widetilde{A} is a positive selfadjoint extension of A that has closed range. Since \widetilde{A} does not have any proper selfadjoint extension, it follows that $\widetilde{A} = \widetilde{A}_N$. According to the first part of the proof there is a nonnegative constant $M \geq 0$ such that

$$(\widetilde{A}h, h) \leq M \cdot \|\widetilde{A}h\|^2$$

for each $h \in \text{dom } \widetilde{A}$. Since \widetilde{A} extends A , we obtain that inequality (4.8) also holds. The proof is complete. \square

The following corollary characterizes those subpositive operators which have any bounded positive selfadjoint extensions with closed range; cf. also [60].

Corollary 4.8. *For a (not necessarily densely defined or closed) positive operator A the following statements are equivalent:*

- (i) *A admits its bounded Krein–von Neumann extension A_N such that A_N has closed range.*
- (ii) *There is a bounded positive selfadjoint extension of A that has closed range.*

(iii) *There exist two nonnegative constants m_1 and m_2 such that*

$$(4.10) \quad \|Ah\|^2 \leq m_1 \cdot (Ah, h) \leq m_2 \cdot \|Ah\|^2$$

holds for all $h \in \text{dom } A$.

Proof. It is clear that (i) implies (ii). If we assume (ii) and \tilde{A} stands for the bounded selfadjoint extension of A that has closed range then the first inequality of (4.10) follows from the Schwarz-inequality by taking $m_1 := \|\tilde{A}\|$ as follows:

$$\|Ah\|^2 = \|\tilde{A}h\|^2 \leq \|\tilde{A}\| \cdot (\tilde{A}h, h) = \|\tilde{A}\| \cdot (Ah, h)$$

for all $h \in \text{dom } A$. The second inequality of (4.10) is obtained from Theorem 4.7. Finally, if we assume that there is a constant $m_1 \geq 0$ such that

$$\|Ah\|^2 \leq m_1 \cdot (Ah, h)$$

for all $x \in \text{dom } A$ then the natural embedding operator J_A of $\text{ran } A \subseteq \mathfrak{H}_A$ into \mathfrak{H} is continuous with norm bound $\sqrt{m_1}$ and hence the Krein-von Neumann extension $J_A^* J_A$ of A is bounded too; see [46] for the details. The second inequality guarantees according to Theorem 4.7 that $J_A^* J_A$ has closed range. \square

As an other application of Theorem 4.7 we give majorization and factorization type characterizations of closed range operators:

Corollary 4.9. *For a densely defined closed operator T between the Hilbert spaces \mathfrak{H} and \mathfrak{K} the following statements are equivalent:*

- (i) *T has closed range.*
- (ii) *There is a nonnegative constant m_T such that*

$$(4.11) \quad \|T^*k\|^2 \leq m_T \cdot \|TT^*k\|^2$$

for all $k \in \text{dom } TT^$.*

- (iii) *There exists a bounded operator S from \mathfrak{H} into \mathfrak{K} such that $T \subseteq TT^*S$.*

Proof. The equivalence of (i) and (ii) follows from Theorem 4.7 by taking $A = T^*T$. If we assume (iii) then the mapping S_0 from $\text{ran } TT^*$ to $\text{ran } T^*$ defined by the relation

$$TT^*k \mapsto T^*k \quad (k \in \text{dom } TT^*)$$

is well defined and gives a continuous linear operator with norm bound m_T so that

$$(4.12) \quad S_0 TT^* \subseteq T^*.$$

Since T is closed (together with TT^*), (iii) follows from (4.12) by taking adjoint and choosing $S = S_0^*$. Finally, the factorization of T by TT^* in (iii) implies range inclusion

$$\text{ran } T \subseteq \text{ran } TT^*$$

which follows (i) by applying Theorem 4.6. \square

We notice that by replacing the role of T and T^* above we obtain further equivalent statements for the closedness of $\text{ran } T$.

We recall the well known fact that a densely defined closed operator T between two Hilbert spaces \mathfrak{H} and \mathfrak{K} that has closed range admits its bounded Moore-Penrose pseudoinverse, i.e. there exists a unique operator $T^\dagger \in \mathcal{B}(\mathfrak{K}, \mathfrak{H})$ satisfying the following properties (cf. [8]):

- (a) $\ker T^\dagger = \text{ran } T^\perp$;
- (b) $T^\dagger T h = P_{\ker T^\perp} h$ for all $h \in \text{dom } T$;
- (c) $T T^\dagger k = P_{\text{ran } T} k$ for all $k \in \mathfrak{K}$.

(For a given closed subspace M its appropriate orthogonal projection is denoted by P_M .) Read more about Moore-Penrose inverses in [8, 9, 10].

In the next theorem we construct the Moore-Penrose pseudoinverse of the Krein-von Neumann extension of a positive operator via factorization used by Sebestyén and Stochel [49], cf. also [60].

Theorem 4.10. *Let A be a positive operator such that A admits its Krein-von Neumann extension A_N which has closed range. Then the operator V_A acting between \mathfrak{H} and \mathfrak{H}_A defined by the following relation*

$$\text{dom } V_A = \{Ah + k : h \in \text{dom } A, k \in \text{ran } A^\perp\}, \quad V_A(Ah + k) = Ah \in \mathfrak{H}_A$$

is densely defined and bounded such that the Moore-Penrose pseudoinverse $(A_N)^\dagger$ of A_N equals $V_A^ V_A^{**}$ and has the norm*

$$(4.13) \quad \|(A_N)^\dagger\| = \inf\{\gamma \geq 0 : (Ah, h) \leq \gamma \cdot \|Ah\|^2 \text{ for all } h \in \text{dom } A\}.$$

Proof. Observe first that our assumptions imply that A_N and J_A^{**} have common ranges. First of all we prove that $\text{ran } A$ is dense in $\text{ran } J_A^{**}$. The range inclusion $\text{ran } A \subseteq \text{ran } J_A^{**}$ is clear. On the other hand, J_A^{**} is the closure of J_A and hence we obtain that

$$\text{ran } J_A^{**} \subseteq \overline{\text{ran } J_A} = \overline{\text{ran } A},$$

thus $\overline{\text{ran } A} = \text{ran } J_A^{**}$, indeed.

It is obvious that $\text{dom } V_A$ is dense in \mathfrak{H} . Since A_N has closed range, Theorem 4.7 implies that there is a constant γ such that

$$\|Ah\|_A^2 = (Ah, h) \leq \gamma \cdot \|Ah\|^2 \leq \gamma \cdot \|Ah + k\|^2$$

for all $h \in \text{dom } A$ and $k \in \text{ran } A^\perp$. Thus V_A is a well defined and continuous operator from \mathfrak{H} into \mathfrak{H}_A with the norm bound $\sqrt{\gamma}$ so that it admits its unique norm preserving continuous extension V_A^{**} to \mathfrak{H} with the same bound. If γ_A denotes the infimum on the

right side of (4.13) then one obtains that $\|V_A\| = \sqrt{\gamma_A}$ as follows:

$$\begin{aligned} \|V_A\|^2 &= \inf \{ \gamma \geq 0 : \langle Ah, Ah \rangle_A \leq \gamma \cdot \|Ah + k\|^2, h \in \text{dom } A, k \in \text{ran } A^\perp \} \\ &= \inf \{ \gamma \geq 0 : (Ah, h) \leq \gamma \cdot \|Ah + k\|^2, h \in \text{dom } A, k \in \text{ran } A^\perp \} \\ &= \inf \{ \gamma \geq 0 : (Ah, h) \leq \gamma \cdot \|Ah\|^2, h \in \text{dom } A \} \\ &= \gamma_A. \end{aligned}$$

In order to show that $V_A^* V_A^{**}$ is the Moore-Penrose pseudoinverse of A_N we are going to prove first that $V_A^{**} = (J_A^{**})^\dagger$, i.e. V_A^{**} and J_A^{**} satisfy properties (a)-(c) above. It is clear that

$$(\text{ran } J_A^{**})^\perp = \text{ran } A^\perp \subseteq \ker V_A^{**}.$$

Conversely, if $h \in \ker V_A^{**}$ and we consider the orthogonal decomposition $h = f + g$ where $f \in \overline{\text{ran } A}$ and $g \in \text{ran } A^\perp$ then there is a sequence $\{f_n\}_{n=1}^\infty$ in $\text{dom } A$ such that $Af_n \rightarrow f$. Since

$$\text{dom } J_A \ni Af_n = V_A^{**}(Af_n + g) \rightarrow V_A^{**}(f + g) = 0 \in \mathfrak{H}_A$$

and

$$J_A(Af_n) = Af_n \rightarrow f \in \mathfrak{H},$$

the fact that J_A is closable implies $f = 0$. Thus $h \in (\text{ran } J_A^{**})^\perp$ and this follows (a). Assume now that $\xi \in \text{dom } J_A^{**}$. By choosing a sequence $\{h_n\}_{n=1}^\infty$ from $\text{dom } A$ such that

$$Ah_n \rightarrow \xi \in \mathfrak{H}_A \quad \text{and} \quad Ah_n \rightarrow J_A^{**}\xi \in \mathfrak{H}$$

we obtain that

$$\xi = \lim_{n \rightarrow \infty} Ah_n = \lim_{n \rightarrow \infty} V_A^{**} J_A^{**}(Ah_n) = V_A^{**} J_A^{**}\xi,$$

and this yields property (b). Finally, assume that $h \in \mathfrak{H}$ and let us consider the orthogonal decomposition $h = f + g$ where $f \in \overline{\text{ran } A}$ and $g \in \text{ran } A^\perp$. By choosing a sequence $\{f_n\}_{n=1}^\infty$ from $\text{dom } A$ with $Af_n \rightarrow f$ we obtain that

$$V_A^{**}(Af_n + g) = Af_n \rightarrow V_A^{**}h \in \mathfrak{H}_A$$

and

$$J_A^{**} V_A^{**}(Af_n + g) = J_A^{**}(Af_n) \rightarrow f \in \mathfrak{H}.$$

However, J_A^{**} is closed so we conclude that $V_A^{**}h \in \text{dom } J_A^{**}$ and $J_A^{**} V_A^{**}h = f = Ph$ where P stands for the orthogonal projection onto $\text{ran } J_A^{**}$. Hence we obtained that $V_A^{**} = (J_A^{**})^\dagger$, indeed. As it is well known, operations $*$ and † "commute". This implies equality $V_A^* = (J_A^*)^\dagger$ and thus properties (a)-(c) for the operator $V_A^* V_A^{**}$ and $J_A^{**} J_A^*$ as follows:

$$\ker V_A^* V_A^{**} = \ker V_A^{**} = (\text{ran } J_A^{**})^\perp = (\text{ran } J_A^{**} J_A^*)^\perp,$$

that yields (a). For $h \in \text{dom } J_A^{**} J_A^*$ we have

$$V_A^* V_A^{**} J_A^{**} J_A^* h = V_A^* I_{\mathfrak{H}_A} J_A^{**} h = Ph,$$

that yields (b) and finally, for all $h \in \mathfrak{H}$ we obtain

$$J_A^{**} J_A^* V_A^* V_A^{**} h = J_A^{**} I_{\mathfrak{H}_A} V_A^{**} h = Ph,$$

that implies property (c). Finally,

$$\|(A_N)^\dagger\| = \|V_A^* V_A^{**}\| = \|V_A^{**}\|^2 = \|V_A\|^2 = \gamma_A$$

according to the C^* -property. The proof is complete. \square

In the next corollary we characterize those subpositive operators whose Krein-von Neumann extension has bounded inverse:

Corollary 4.11. *Let A be a positive linear operator in a Hilbert space \mathfrak{H} such that $\mathcal{D}_*(A)$ is dense. The following statements are equivalent:*

- (i) *The Krein-von Neumann extension A_N of A has bounded inverse;*
- (ii) *The range $\text{ran } A$ of A is dense in \mathfrak{H} and there is a constant $\gamma > 0$ such that*

$$(4.14) \quad (Af, f) \leq \gamma \cdot \|Af\|^2$$

for all $f \in \text{dom } A$.

Moreover, if $V_A : H \supseteq \text{ran } A \rightarrow \mathfrak{H}_A$ is given by the formula

$$V_A(Ah) = Ah \in \mathfrak{H}_A \quad (h \in \text{dom } A)$$

then $(A_N)^{-1} = V_A^* V_A^{**}$ and

$$\|(A_N)^{-1}\| = \inf\{\gamma \geq 0 : (Af, f) \leq \gamma \cdot \|Af\|^2 \text{ for all } f \in \text{dom } A\}.$$

Proof. Straightforward from the previous theorem. \square

4.4. Extremal extensions with closed range

A positive selfadjoint extension \tilde{A} of a given positive operator A is said to be *extremal* if it satisfies

$$(4.15) \quad \inf \{ (\tilde{A}(h - f), h - f) : f \in \text{dom } A \} = 0$$

for all $h \in \text{dom } \tilde{A}$. The notion of extremal extensions was introduced by Arlinskii and Tsekanovskii [6]. The characterization of all extremal extensions \tilde{A} of a densely defined positive operator A was given by Arlinskii, Hassi, Sebestyén and de Snoo [5] by showing that there exists a dense linear manifold \mathfrak{L} containing $\text{dom } A$ and contained by $\mathcal{D}_*(A)$ such that

$$(4.16) \quad R^* R^{**} = \tilde{A},$$

where R denotes the restriction of J_A^* to \mathfrak{L} , see Theorem 4.4 of [5]. Moreover, \mathfrak{L} can also be chosen so that R is closed, namely by taking $\mathfrak{L} := \text{dom } \tilde{A}^{1/2}$. Conversely, according to the density of $\text{ran } A$ of \mathfrak{H}_A , the positive selfadjoint extension $A_{\mathfrak{L}} := R^*R^{**}$ is extremal whenever \mathfrak{L} is a dense linear manifold of \mathfrak{H} satisfying $\text{dom } A \subseteq \mathfrak{L} \subseteq \mathcal{D}_*(A)$, see Proposition 4.1 of [5].

By choosing $\mathfrak{L} := \mathcal{D}_*(A)$ one obtains that $R^*R^{**} = J_A^*J_A$, i.e. the Krein–von Neumann extension of A is extremal. If we require in addition that A is densely defined then $\text{dom } A$ offers itself as another plausible alternative for \mathfrak{L} : the positive selfadjoint operator R^*R^{**} turns out to coincide then with the Friedrichs extension A_F , the largest among all positive selfadjoint extensions of A in that case, see [39, 40]. We notice that the extra assumption that $\text{dom } A$ is dense is also necessary (and not only sufficient) to guarantee the existence of the Friedrichs extension of A .

The aim of this section is on one hand to characterize those densely defined positive operators whose Friedrichs extension A_F has closed range. But first we need the following lemma describing the domain and the range of the square root of A_F :

Lemma 4.12. *Let A be a densely defined positive operator in a Hilbert space \mathfrak{H} and let A_F denote the Friedrichs extension of A . Then \mathcal{D} and \mathcal{R} , the domain and the range of the square root of A_F , respectively, can be described as follows:*

$$(4.17) \quad \begin{cases} \mathcal{D} = \{h \in H : \exists \{f_n\}_{n=1}^{\infty} \subseteq \text{dom } A, f_n \rightarrow h, (A(f_n - f_m), f_n - f_m) \rightarrow 0\}, \\ \mathcal{R} = \{g \in H : \exists m_g \geq 0, |(f, g)|^2 \leq m_g \cdot (Af, f) \text{ for all } f \in \text{dom } A\}. \end{cases}$$

Proof. Let R denote the restriction of J_A^* to $\text{dom } A$. Then R is a densely defined closable operator from \mathfrak{H} into \mathfrak{H}_A so that the positive selfadjoint operator R^*R^{**} equals the Friedrichs extension of A , cf. [40, 39] or [48]. Thus, according to the characteristic extension property of J_A^* in (5.4) we obtain that

$$\begin{aligned} \text{dom } A_F^{1/2} &= \text{dom } R^{**} \\ &= \{h \in \mathfrak{H} : \exists \{f_n\}_{n=1}^{\infty} \subseteq \text{dom } R, f_n \rightarrow h, R(f_n - f_m) \rightarrow 0\} \\ &= \{h \in \mathfrak{H} : \exists \{f_n\}_{n=1}^{\infty} \subseteq \text{dom } A, f_n \rightarrow h, A(f_n - f_m) \rightarrow 0 \text{ in } \mathfrak{H}_A\} \\ &= \{h \in \mathfrak{H} : \exists \{f_n\}_{n=1}^{\infty} \subseteq \text{dom } A, f_n \rightarrow h, (A(f_n - f_m), f_n - f_m) \rightarrow 0\} \\ &= \mathcal{D}, \end{aligned}$$

and on the other hand, using the range characterization of Sebestyén [44] for the adjoint operator it follows that

$$\begin{aligned} \text{ran } A_F^{1/2} &= \text{ran } R^* \\ &= \{g \in \mathfrak{H} : \exists m_g \geq 0, |(f, g)|^2 \leq m_g \cdot \langle Rf, Rf \rangle_A \text{ for all } f \in \text{dom } A\} \\ &= \{g \in \mathfrak{H} : \exists m_g \geq 0, |(f, g)|^2 \leq m_g \cdot (Af, f) \text{ for all } f \in \text{dom } A\} \\ &= \mathcal{R}. \end{aligned}$$

The proof is complete. \square

The next result characterizes those positive operators whose Friedrichs extension has closed range:

Theorem 4.13. *Let A be a densely defined positive operator in a Hilbert space \mathfrak{H} . The following two assertions are equivalent:*

- (i) *The Friedrichs extension of A has closed range;*
- (ii) *There is a constant $\gamma > 0$ such that*

$$(4.18) \quad \sup \{ |(Af, h)|^2 : f \in \text{dom } A, (Af, f) \leq 1 \} \geq \gamma \cdot \|h\|^2$$

for all $h \in \mathcal{D} \cap \overline{\mathcal{R}}$.

Here the symbols \mathcal{D} and \mathcal{R} stand for the linear manifolds introduced in (4.17).

Proof. We recall that a densely defined closed linear operator T between two Hilbert spaces has closed range if and only if there is a constant $\gamma > 0$ such that

$$\|Th\|^2 \geq \gamma \cdot \|h\|^2 \quad \text{for all } h \in \text{dom } T \cap (\ker T^*)^\perp.$$

Thus, according to Theorem 4.6 and Lemma 4.12, A_F has closed range if and only if $A_F^{1/2}$ has, i.e. when

$$\|A_F^{1/2}h\|^2 \geq \gamma \cdot \|h\|^2 \quad \text{for all } h \in \mathcal{D} \cap \overline{\mathcal{R}}.$$

Using the density of $\text{ran } A$ of the model Hilbert space \mathfrak{H}_A , the left side of the above inequality can be expressed as it is claimed in (4.18) as follows:

$$\begin{aligned} \|A_F^{1/2}h\|^2 &= \langle R^{**}h, R^{**}h \rangle_A \\ &= \sup \{ |\langle R^{**}h, Af \rangle_A|^2 : f \in \text{dom } A, \langle Af, Af \rangle_A \leq 1 \} \\ &= \sup \{ |(h, J_A(Af))|^2 : f \in \text{dom } A, (Af, f) \leq 1 \} \\ &= \sup \{ |(h, Af)|^2 : f \in \text{dom } A, (Af, f) \leq 1 \}, \end{aligned}$$

and this completes the proof. \square

Corollary 4.14. *Let A be a densely defined positive operator in a Hilbert space \mathfrak{H} . The following statements are equivalent:*

- (i) *The Friedrichs extension of A has bounded inverse;*
- (ii) *There is a constant $\gamma > 0$ such that*

$$(4.19) \quad (Af, f) \geq \gamma \cdot \|f\|^2$$

for all $f \in \text{dom } A$.

Proof. If the Friedrichs extension A_F of A is boundedly invertible then its square root $A_F^{1/2}$ must be bounded from below with some positive lower bound γ , thus (i) implies (ii). If R stands for the restriction of J_A^* to $\text{dom } A$ then from inequality (4.19) we conclude that

$$\langle Rf, Rf \rangle_A = \langle J_A^* f, J_A^* f \rangle_A = \langle Af, Af \rangle_A = \langle Af, f \rangle \geq \gamma \cdot \|f\|^2$$

holds for each $f \in \text{dom } R$ and thus its closure R^{**} is bounded from below. This implies that R^{**} has closed range. On the other hand, $\text{ran } R^{**}$ contains the dense linear manifold $\text{ran } A$ of \mathfrak{H}_A , i.e. R^{**} has bounded inverse. Consequently, the Friedrichs extension $R^* R^{**}$ of A is also boundedly invertible. \square

Lemma 4.15. *Assume that A is a (not necessarily densely defined) positive operator, \mathfrak{L} and \mathfrak{M} are dense linear subspaces of \mathfrak{H} satisfying*

$$\text{dom } A \subseteq \mathfrak{L} \subseteq \mathfrak{M} \subseteq \mathcal{D}_*(A).$$

If R and Q denote the restrictions of J_A^ to \mathfrak{L} and \mathfrak{M} , respectively, then*

- (a) $Q^* Q^{**} \leq R^* R^{**}$,
- (b) $R^{**} R^* \leq Q^{**} Q^*$.

Proof. Since $\text{dom } R = \mathfrak{L}$ is contained by $\text{dom } Q = \mathfrak{M}$, one obtains that Q^{**} extends R^{**} . Thus on one hand

$$\text{dom}(R^* R^{**})^{1/2} = \text{dom } R^{**} \subseteq \text{dom } Q^{**} = \text{dom}(Q^* Q^{**})^{1/2}$$

and for each $h \in \text{dom}(R^* R^{**})^{1/2}$

$$\|(R^* R^{**})^{1/2} h\|^2 = \langle R^{**} h, R^{**} h \rangle_A = \langle Q^{**} h, Q^{**} h \rangle_A = \|(Q^* Q^{**})^{1/2} h\|^2$$

on the other hand. This follows part (a). The statement of (b) can be proved in a completely analogous way. \square

If \tilde{A} and \tilde{B} are now extremal extensions of the positive operator A then by applying of the preceding lemma one obtains that assertion $\tilde{A} \leq \tilde{B}$ is equivalent to the inclusion $\text{dom } \tilde{B}^{1/2} \subseteq \text{dom } \tilde{A}^{1/2}$. Therefore, $\tilde{A} = \tilde{B}$ if and only if $\text{dom } \tilde{B}^{1/2} = \text{dom } \tilde{A}^{1/2}$. The following lemma gives another useful equivalent statement:

Lemma 4.16. *If \tilde{A} and \tilde{B} are extremal extensions of a given positive operator A then the following assertions are equivalent*

- (i) $\tilde{A} = \tilde{B}$
- (ii) $\tilde{A} \leq \tilde{B}$ (or $\tilde{B} \leq \tilde{A}$) and $\text{ran } \tilde{A}^{1/2} = \text{ran } \tilde{B}^{1/2}$.

Proof. By letting $\mathfrak{L} := \text{dom } \tilde{A}^{1/2}$ and $\mathfrak{M} := \text{dom } \tilde{B}^{1/2}$ we obtain

$$R^* R = \tilde{A} \quad \text{and} \quad Q^* Q = \tilde{B},$$

where R and Q are the restrictions of J_A^* to the dense linear manifolds \mathfrak{L} and \mathfrak{M} , respectively. If we assume equality $\widetilde{\text{ran}} \tilde{A}^{1/2} = \text{ran} \tilde{B}^{1/2}$ on the ranges and for instance $\tilde{B} \leq \tilde{A}$, then Q extends R . Hence we have $Q^* \subseteq R^*$ on the adjoint operators so that they have common ranges. On the other hand, since both $\text{ran } R$ and $\text{ran } Q$ are dense in the model Hilbert space \mathfrak{H}_A , we also have

$$\ker R^* = \{0\} = \ker Q^*.$$

A simple argument shows that then $R^* = Q^*$. Since the operators R and Q are closed, we find that our original operators \tilde{A} and \tilde{B} are also equal. \square

Theorem 4.17. *Assume that A is a positive operator in a Hilbert space \mathfrak{H} with extremal extensions \tilde{A} and \tilde{B} where $\tilde{B} \leq \tilde{A}$. If \tilde{A} has closed range then \tilde{B} has closed range too.*

Proof. Let us use the notions of the proof of Lemma 4.16. Since $\text{ran } RR^*$ is closed by Theorem 4.6 and contains the dense linear manifold $\text{ran } A$ it follows that RR^* is boundedly invertible on the model Hilbert space \mathfrak{H}_A . According to part (b) of Lemma 4.15 we obtain that the inverse of QQ^* is also bounded and therefore, according again to Theorem 4.6, $\tilde{B} = Q^*Q$ has closed range. \square

Taking into account the above theorem, the Krein-von Neumann extension of A always has closed range whenever an extremal extension of A exists the range of which is closed. On the other hand, if $\text{dom } A$ is assumed to be dense then the Friedrichs extension of A is also extremal. This leads us to the following result:

Corollary 4.18. *Assume that A is a densely defined positive operator in a Hilbert space \mathfrak{H} . The following statements are equivalent:*

- (i) *The Friedrichs extension of A has closed range;*
- (ii) *All extremal extensions \tilde{A} of A have closed range.*

In the last result of this chapter some characterizations of essentially selfadjointness of positive operators are presented. The notions $(A + I)_N$ and $(A + I)_F$ below stand for the Krein-von Neumann and the Friedrichs extensions of $A + I$, respectively.

Theorem 4.19. *Let A be a densely defined positive operator in a Hilbert space \mathfrak{H} . The following statements are equivalent:*

- (i) *A is essentially selfadjoint;*
- (ii) *$A + I$ admits a unique positive selfadjoint extension, i.e. $(A + I)_N = (A + I)_F$;*
- (iii) *The range of $A + I$ is dense in \mathfrak{H} ;*
- (iv) *$(A + I)_N = A_N + I$;*
- (v) *$(A + I)_N$ has bounded inverse.*

Proof. According to identity $(A + I)^{**} = A^{**} + I$, (i) implies all of (ii)-(v). Now first we show that (ii) implies (i). Since the densely defined positive operator $A + I$ is bounded

from below with lower bound $\gamma = 1$, Corollary 4.14 implies that its Friedrichs extension $(A + I)_F$ has bounded inverse. Assumption (ii) and Corollary 4.11 imply then that the range of $A + I$ is dense and, by noticing that $((A + I)_N)_N = (A + I)_N$, there is a constant $\gamma > 0$ such that

$$((A + I)_N h, h) \leq \gamma \cdot \|(A + I)_N h\|^2$$

holds for all $h \in \text{dom}(A + I)_N$. Since $(A + I)^{**} \subseteq (A + I)_N$ we conclude that for all $h \in \text{dom}(I + A)^{**}$

$$\|h\|^2 \leq ((A + I)^{**} h, h) \leq \gamma \cdot \|(A + I)^{**} h\|^2.$$

This means that $(A + I)^{**}$ is a closed operator which is bounded from below, i.e. its range is closed. Since it contains the dense linear manifold $\text{ran}(A + I)$, it follows that $(A + I)^{**} = A^{**} + I$ is surjective and hence selfadjoint. Therefore, A is essentially selfadjoint. Assume now (iii) and prove (ii). According to Lemma 4.16 it is enough to show that

$$(4.20) \quad \text{ran}(A + I)_N^{1/2} = \text{ran}(A + I)_F^{1/2}.$$

According to the first part of the proof, the Friedrichs extension of $A + I$ is surjective. In particular, it has closed range. Thus the range of $(A + I)_N$ is also closed by Corollary 4.18, i.e. it is equal to \mathfrak{H} . This follows equality (4.20), as required. Assumptions (iv) and (v) mean among others that $(A + I)_N$ is surjective. Consequently, (4.20) is satisfied and the preceding argument shows that both of (iv) and (v) imply (ii). The proof is complete. \square

Remark 4.20. As it is well known, for any densely defined positive operator A the identity $(A + I)_F = A_F + I$ is valid, see Theorem 10 of [32]. Nevertheless, the preceding theorem says that $(A + I)_N$ does not equal $A_N + I$ in general, cf. also [3]. However, taking into account Corollary 4.14 and Corollary 4.18, the range of $(I + A)_N$ is always a closed subspace of \mathfrak{H} and this containing is proper unless A is essentially selfadjoint.

On form sums of positive operators

If A is a positive selfadjoint operator in a Hilbert space \mathfrak{H} then the following relation

$$(5.1) \quad \mathfrak{t}_A(f, g) := (A^{1/2}f, A^{1/2}g), \quad f, g \in \text{dom } A^{1/2},$$

defines a positive definite sesquilinear form over $\text{dom } A^{1/2}$ which is closed in the classical sense (cf. Kato [30, IV §5]): if $\{f_n\}_{n=1}^\infty$ is a sequence from $\text{dom } A^{1/2}$ so that $f_n \rightarrow f$ and that $\{\mathfrak{t}_A(f_n, f_n)\}_{n=1}^\infty$ is convergent then $f \in \text{dom } \mathfrak{t}_A$ and $\mathfrak{t}_A(f_n - f, f_n - f) \rightarrow 0$. If another positive selfadjoint operator B in \mathfrak{H} is given and \mathfrak{t}_B stands for its associated form then $\mathfrak{t}_A + \mathfrak{t}_B$ on $\text{dom } A^{1/2} \cap \text{dom } B^{1/2}$ is also closed. In general case it also can happen that the domain of $\mathfrak{t}_A + \mathfrak{t}_B$ is just the trivial subspace. But if $\text{dom } A^{1/2} \cap \text{dom } B^{1/2}$ is assumed to be dense then the representation theorem yields a unique positive selfadjoint operator C with the characteristic properties $\text{dom } C \subset \text{dom } A^{1/2} \cap \text{dom } B^{1/2}$ and

$$(Cf, g) = (A^{1/2}f, A^{1/2}g) + (B^{1/2}f, B^{1/2}g),$$

holds for $f \in \text{dom } C$ and $g \in \text{dom } A^{1/2} \cap \text{dom } B^{1/2}$. C is used to be called the *form sum* of the positive operators A and B and is usually denoted by $A \dot{+} B$. In the case when $\text{dom } A^{1/2} \cap \text{dom } B^{1/2}$ is not dense, the form sum of A and B can only be defined as a linear relation, i.e. as a multivalued operator, cf. [21, 22].

Let $\mathscr{D}_*(A), \mathfrak{H}_A, J_A$, etc., be defined just as in the introduction of Chapter , and let $A_N := J_A^{**} J_A^*$ be the Krein–von Neumann extension of the positive operator A . Recall that A_N exists if and only if the $\mathscr{D}_*(A)$ is dense in \mathfrak{H} . The well known identity $\text{dom}(J_A^{**} J_A^*)^{1/2} = \text{dom } J_A^*$ and

$$(5.2) \quad \text{dom } J_A^* = \mathscr{D}_*(A)$$

imply that the characteristic set $\mathscr{D}_*(A)$ coincides with the domain of the square $A_N^{1/2}$ root of A_N . Thus, if A and B are positive operators such that $\mathscr{D}_*(A) \cap \mathscr{D}_*(B)$ is dense then the form sum $A_N \dot{+} B_N$ of the Krein–von Neumann extensions exists. The main purpose of the this chapter is to characterize the domain, the kernel and the range spaces of the form sum $A_N \dot{+} B_N$ of the Krein–von Neumann extensions of the positive operators A and B , under the only assumption that it exist, i.e. when the linear manifold $\mathscr{D}_*(A) \cap \mathscr{D}_*(B)$ is dense in \mathfrak{H} .

The treatment used throughout the paper is essentially based on the method of Sebestyén and Stochel [49] above for giving the Krein–von Neumann extension of a non-negative operator, and on the form sum construction of Farkas and Matolcsi [16]. The

main idea by characterizing the domain, kernel and range of the form sum is taken from [48] of Sebestyén and Sikolya. The approach in the last section by investigating the conditions guaranteeing the closedness of the range space $\text{ran}(A_N \dot{+} B_N)$ generalizes some results of [61].

5.1. The form sum construction

Assume that A and B are positive selfadjoint operators such that $\mathcal{D}_*(A) \cap \mathcal{D}_*(B)$ is dense in \mathfrak{H} . Taking into account of the above comments, this expresses that the form sum of the positive selfadjoint operators A_N and B_N exists. The following construction for giving the operator $A_N \dot{+} B_N$ arises from B. Farkas and M. Matolcsi [16]. We present here a short proof of their result via the following lemma; cf. also [21]:

Lemma 5.1. *Let \mathfrak{H}_1 , \mathfrak{H}_2 and \mathfrak{K} be Hilbert spaces and let $S : \mathfrak{H}_1 \rightarrow \mathfrak{K}$, $T : \mathfrak{H}_2 \rightarrow \mathfrak{K}$ be densely defined linear operators. Let $J : \mathfrak{H}_1 \times \mathfrak{H}_2 \rightarrow \mathfrak{K}$ be the mapping defined by the following correspondence:*

$$(5.3) \quad \{f, g\} \mapsto Sf + Tg, \quad f \in \text{dom } S, g \in \text{dom } T.$$

Then $\text{dom } J^ = \text{dom } S^* \cap \text{dom } T^*$ and*

$$(5.4) \quad J^*k = \{S^*k, T^*k\}, \quad k \in \text{dom } J^*.$$

If in addition, $\text{dom } S^ \cap \text{dom } T^*$ is assumed to be dense then $J^{**}J^*$ equals the form sum of the selfadjoint operators $S^{**}S^*$ and $T^{**}T^*$:*

$$(5.5) \quad J^{**}J^* = S^{**}S^* \dot{+} T^{**}T^*.$$

Proof. If $k \in \text{dom } S^* \cap \text{dom } T^*$ then for all $f \in \text{dom } S$ and $g \in \text{dom } T$ we have

$$\begin{aligned} |(Sf + Tg, k)| &\leq |(f, S^*k)| + |(g, T^*k)| \\ &\leq \max\{\|S^*k\|, \|T^*k\|\} \cdot (\|f\| + \|g\|) \\ &\leq \sqrt{2} \max\{\|S^*k\|, \|T^*k\|\} \cdot \|\{f, g\}\|, \end{aligned}$$

therefore $k \in \text{dom } J^*$. Conversely, if $k \in \text{dom } J^*$ then for all $f \in \text{dom } S$ we have $\{f, 0\} \in \text{dom } J$ and that

$$|(Sf, k)| = |(J\{f, 0\}, k)| \leq \|J^*k\| \cdot \|\{f, 0\}\| = \|J^*k\| \cdot \|f\|,$$

that is, $k \in \text{dom } S^*$. We can prove in completely analogous way that $k \in \text{dom } T^*$. Hence we have $\text{dom } J^* = \text{dom } S^* \cap \text{dom } T^*$, indeed. To prove identity (5.4) let $k \in \text{dom } J^*$. For each $f \in \text{dom } S$ and $g \in \text{dom } T$ one obtains that

$$(J^*k - \{S^*k, T^*k\}, \{f, g\}) = (k, Sf + Tg) - (k, Sf) - (k, Tg) = 0,$$

thus $J^*k - \{S^*k, T^*k\}$ is orthogonal to the dense subset $\text{dom } S \times \text{dom } T$ of $\mathfrak{H}_1 \times \mathfrak{H}_2$ which yields (5.4).

Finally, if we assume in addition that $\text{dom } T^* \cap \text{dom } S^*$ is dense, then, according to the first part of the proof, each of operators J , S and T are closable so that $A := S^{**}S^*$, $B := T^{**}T^*$ and $J^{**}J^*$ are positive selfadjoint operators in the Hilbert space \mathfrak{K} due to von Neumann's classical theorem. Since $\text{dom } A^{1/2} = \text{dom } S^*$ and $\text{dom } B^{1/2} = \text{dom } T^*$, it follows that

$$\text{dom } J^{**}J^* \subseteq \text{dom } J^* = \text{dom } A^{1/2} \cap \text{dom } B^{1/2}.$$

According to identity (5.4), for all $f \in \text{dom } J^{**}J^*$ and $g \in \text{dom } A^{1/2} \cap \text{dom } B^{1/2}$

$$\begin{aligned} (J^{**}J^*f, g) &= (J^*f, J^*g) = (\{S^*f, T^*f\}, \{S^*g, T^*g\}) \\ &= (S^*f, S^*g) + (T^*f, T^*g) \\ &= (A^{1/2}f, A^{1/2}g) + (B^{1/2}f, B^{1/2}g). \end{aligned}$$

Therefore, $J^{**}J^* = A \dot{+} B$, indeed. \square

As a consequence we have the following generalization of [17, Theorem 2.2].

Corollary 5.2. *Assume that $S : \mathfrak{H}_1 \rightarrow \mathfrak{K}$ and $T : \mathfrak{H}_2 \rightarrow \mathfrak{K}$ are densely defined and closed operators between Hilbert spaces such that $\text{dom } S^* \cap \text{dom } T^*$ is dense in \mathfrak{K} . Then*

$$\text{ran } S + \text{ran } T \subseteq \text{ran}(SS^* \dot{+} TT^*)^{1/2}.$$

If S and T are everywhere defined continuous operators, the above range inclusion becomes equality.

Proof. Let J stand for the operator introduced in (5.3). Then J is closable and

$$(5.6) \quad \text{ran } S + \text{ran } T = \text{ran } J \subseteq \text{ran } J^{**} = \text{ran}(J^{**}J^*)^{1/2} = \text{ran}(SS^* \dot{+} TT^*)^{1/2}$$

according to Lemma 5.1. If S and T are bounded operators then J is defined on the whole of $\mathfrak{H}_1 \times \mathfrak{H}_2$ so that J coincides with its closure J^{**} . Thus (5.6) yields the desired identity. \square

Theorem 5.3. *Let A and B be positive operators such that $\mathcal{D}_*(A) \cap \mathcal{D}_*(B)$ is dense. Let $\mathfrak{H}_A, \mathfrak{H}_B$ and J_A, J_B denote the appropriate auxiliary Hilbert spaces and operators. Let $J : \mathfrak{H}_A \times \mathfrak{H}_B \rightarrow \mathfrak{H}$ be stand for the following linear operator*

$$(5.7) \quad \{Af, Bg\} \mapsto Af + Bg, \quad f \in \text{dom } A, g \in \text{dom } B.$$

*Then the form sum of A_N and B_N equals $J^{**}J^*$.*

Proof. It is an immediate consequence of Lemma 5.1 by choosing Hilbert spaces $\mathfrak{H}_1 := \mathfrak{H}_A, \mathfrak{H}_2 := \mathfrak{H}_B, \mathfrak{K} := \mathfrak{H}$ and operators $S := J_A, T := J_B$. \square

We notice that if $A = S^*S$ and $B = T^*T$ with any densely defined, not necessarily closable or closed operators S and T then A and B admit their Krein–von Neumann extensions (see [53]). Moreover, in that case the density of $\text{dom } S \cap \text{dom } T$ guarantees the existence of the form sum $A_N \dot{+} B_N$.

5.2. Domain, kernel and range characterizations

Via the construction given in Theorem 5.3 we are able to characterize the domain, the kernel and the range of the form sum $A_N \dot{+} B_N$ and of its square root $(A_N \dot{+} B_N)^{1/2}$. These characterizations are somewhat simpler in case of the square root, so we start with it.

Theorem 5.4. *Let A and B be positive operators such that $\mathcal{D}_*(A) \cap \mathcal{D}_*(B)$ is dense in \mathfrak{H} . Then*

- (a) $\text{dom}(A_N \dot{+} B_N)^{1/2} = \mathcal{D}_*(A) \cap \mathcal{D}_*(B)$.
- (b) $\ker(A_N \dot{+} B_N)^{1/2} = \{\text{ran } A + \text{ran } B\}^\perp$.

Proof. According to Theorem 5.3, Lemma 5.1 and identity (5.2) we obtain that

$$\text{dom}(A_N \dot{+} B_N)^{1/2} = \text{dom } J^* = \text{dom } J_A^* \cap \text{dom } J_B^* = \mathcal{D}_*(A) \cap \mathcal{D}_*(B),$$

which shows (a). Statement (b) is obtained from the following identities

$$\ker(A_N \dot{+} B_N)^{1/2} = \ker J^* = \{\text{ran } J\}^\perp = \{\text{ran } A + \text{ran } B\}^\perp.$$

□

Theorem 5.5. *Let A and B be positive operators such that $\mathcal{D}_*(A) \cap \mathcal{D}_*(B)$ is dense in \mathfrak{H} , and let $y \in \mathfrak{H}$. The following statements are equivalent:*

- (i) $y \in \text{ran}(A_N \dot{+} B_N)^{1/2}$.
- (ii) *There is a constant $m_y \geq 0$ such that for all $h \in \mathcal{D}_*(A) \cap \mathcal{D}_*(B)$ the following inequality holds*

$$|(y, h)|^2 \leq m_y \cdot (M_A(h) + M_B(h)),$$

where $M_A(h)$ and $M_B(h)$ are defined as follows

$$(5.8) \quad \begin{cases} M_A(h) := \sup \{|(Af, h)|^2 : f \in \text{dom } A, (Af, f) \leq 1\}, \\ M_B(h) := \sup \{|(Bg, h)|^2 : g \in \text{dom } B, (Bg, g) \leq 1\}. \end{cases}$$

- (iii) *There are two sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ from $\text{dom } A$ and $\text{dom } B$, respectively, with $(A(f_n - f_m), f_n - f_m) \rightarrow 0$, $(B(g_n - g_m), g_n - g_m) \rightarrow 0$ and $Af_n + Bg_n \rightarrow y$.*

Proof. The equivalence of (i) and (ii) is obtained by using [44, Theorem 1] describing the range of the adjoint operator as follows:

$$\begin{aligned} \text{ran}(A_N \dot{+} B_N)^{1/2} &= \text{ran } J^{**} = \text{ran}(J^*)^* \\ &= \{y \in \mathfrak{H} : |(y, h)| \leq m_y \cdot \|J^*h\|_{\mathfrak{H}_A \times \mathfrak{H}_B} \text{ for all } h \in \text{dom } J^*\}. \end{aligned}$$

According to identity (5.4) of Lemma 5.1, $J^*h = \{J_A^*h, J_B^*h\}$ for any $h \in \text{dom } J^* = \mathcal{D}_*(A) \cap \mathcal{D}_*(B)$. Using the density of $\text{ran } A$ and $\text{ran } B$ in the auxiliary Hilbert spaces \mathfrak{H}_A

and \mathfrak{H}_B , respectively, one obtains that

$$\begin{aligned}\|J_A^*h\|_A^2 &:= \langle J_A^*h, J_A^*h \rangle_A = \sup \left\{ |\langle J_A^*h, Af \rangle_A|^2 : f \in \text{dom } A, \langle Af, Af \rangle_A \leq 1 \right\} \\ &= \sup \left\{ |(h, Af)|^2 : f \in \text{dom } A, \langle Af, f \rangle \leq 1 \right\},\end{aligned}$$

and analogously that

$$\|J_B^*h\|_B^2 := \langle J_B^*h, J_B^*h \rangle_B = \sup \left\{ |(h, Bg)|^2 : g \in \text{dom } B, \langle Bg, g \rangle \leq 1 \right\}.$$

In view of these observations,

$$(5.9) \quad \|J_A^*h\|_A^2 + \|J_B^*h\|_B^2 = M_A(h) + M_B(h),$$

which proves the desired equivalence.

Now we prove the equivalence of (i) and (iii). By referring again to range equality $\text{ran}(A_N \dot{+} B_N)^{1/2} = \text{ran } J^{**}$ and using the fact that J^{**} is just the closure of J , we conclude that a vector $y \in \mathfrak{H}$ is in the range of $(A_N \dot{+} B_N)^{1/2}$ if and only if there is a sequence $\{f_n, g_n\}_{n=1}^\infty$ from $\text{dom } A \times \text{dom } B$ such that $\{Af_n, Bg_n\}_{n=1}^\infty$ is convergent in the product Hilbert space $\mathfrak{H}_A \times \mathfrak{H}_B$ so that $J\{Af_n, Bg_n\} = Af_n + Bg_n \rightarrow y$. This is just the statement of assertion (iii). \square

Since the kernel of a positive selfadjoint operator and of its square root coincide, Theorem 5.4 yields

$$\ker(A_N \dot{+} B_N) = \{\text{ran } A + \text{ran } B\}^\perp.$$

The following two results describe the domain and the range of the form sum $A_N \dot{+} B_N$.

Theorem 5.6. *Let A and B be positive operators such that $\mathcal{D}_*(A) \cap \mathcal{D}_*(B)$ is dense in \mathfrak{H} , and let $h \in \mathfrak{H}$. The following statements are equivalent:*

- (i) $h \in \text{dom}(A_N \dot{+} B_N)$.
- (ii) $h \in \mathcal{D}_*(A) \cap \mathcal{D}_*(B)$ and there are two sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ from $\text{dom } A$ and $\text{dom } B$, respectively, such that
 - (a) $\sup_{n \in \mathbb{N}} \langle Af_n, f_n \rangle < \infty$ and $\langle f_n, Af \rangle \rightarrow \langle h, Af \rangle$ for all $f \in \text{dom } A$;
 - (b) $\sup_{n \in \mathbb{N}} \langle Bg_n, g_n \rangle < \infty$ and $\langle g_n, Bg \rangle \rightarrow \langle h, Bg \rangle$ for all $g \in \text{dom } B$;
 - (c) $\{Af_n + Bg_n\}_{n=1}^\infty$ converges.

Proof. Due to the form sum construction in Theorem 5.3 we have

$$\begin{aligned}\text{dom}(A_N \dot{+} B_N) &= \{h \in \text{dom } J^* : J^*h \in \text{dom } J^{**}\} \\ &= \{h \in \mathcal{D}_*(A) \cap \mathcal{D}_*(B) : \{J_A^*h, J_B^*h\} \in \text{dom } J^{**}\}.\end{aligned}$$

By putting a vector $h \in \mathcal{D}_*(A) \cap \mathcal{D}_*(B)$, one obtains that $\{J_A^*h, J_B^*h\}$ is in domain of J^{**} if and only if there is a sequence $\{f_n, g_n\}_{n=1}^\infty$ from $\text{dom } A \times \text{dom } B$ such that $\{Af_n, Bg_n\} \rightarrow \{J_A^*h, J_B^*h\}$ in $\mathfrak{H}_A \times \mathfrak{H}_B$ and that $J\{Af_n, Bg_n\} = Af_n + Bg_n$ converges in \mathfrak{H} . In order that a sequence $\{Af_n, Bg_n\}_{n=1}^\infty$ converge to $\{J_A^*h, J_B^*h\}$ it is necessary and

sufficient that both $\{Af_n\}_{n=1}^\infty$ and $\{Bg_n\}_{n=1}^\infty$ are bounded in the corresponding auxiliary Hilbert spaces and that $\{Af_n\}_{n=1}^\infty$ converges weakly to J_A^*h on $\text{ran } A$ and that $\{Bg_n\}_{n=1}^\infty$ converges weakly to J_B^*h on $\text{ran } B$ where the sufficiency is due to the density of the appropriate linear manifolds. In other words, for any $f \in \text{dom } A$

$$\langle Af_n - J_A^*h, Af \rangle_A = (f_n - h, Af) \rightarrow 0 \quad \text{and} \quad \sup_{n \in \mathbb{N}} (Af_n, f_n) < \infty,$$

and, respectively, for any $g \in \text{dom } B$

$$(g_n - h, Bg) \rightarrow 0 \quad \text{and} \quad \sup_{n \in \mathbb{N}} (Bg_n, g_n) < \infty.$$

According to the reasoning above, the desired equivalence is clear. \square

Theorem 5.7. *Let A and B be positive operators such that $\mathfrak{D}_*(A) \cap \mathfrak{D}_*(B)$ is dense in \mathfrak{H} , and let $y \in \mathfrak{H}$. The following statements are equivalent:*

- (i) $y \in \text{ran}(A_N \dot{+} B_N)$.
- (ii) *There are two sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ from $\text{dom } A$ and $\text{dom } B$, respectively, such that $(A(f_n - f_m), f_n - f_m) \rightarrow 0$, $(B(g_n - g_m), g_n - g_m) \rightarrow 0$, $Af_n + Bg_n \rightarrow y$, and that for any $f \in \text{dom } A$ and $g \in \text{dom } B$*

$$\limsup_{n \rightarrow \infty} |(Af_n, f) + (Bg_n, g)| \leq m_y \|Af + Bg\|,$$

where m_y is a constant depending only on y .

Proof. In view of our construction for the form sum we find that

$$\text{ran}(A_N \dot{+} B_N) = \{J^{**}\{\xi, \eta\} : \{\xi, \eta\} \in \text{ran } J^* \cap \text{dom } J^{**}\}.$$

On one hand, $\{\xi, \eta\} \in \text{ran } J^*$ holds if and only if there is a constant $m_{\xi, \eta}$ such that for any $f \in \text{dom } A$ and $g \in \text{dom } B$

$$\left| \langle \{\xi, \eta\}, \{Af, Bg\} \rangle_{\mathfrak{H}_A \times \mathfrak{H}_B} \right| \leq m_{\xi, \eta} \|J\{Af, Bg\}\|,$$

due to [44, Theorem 1]. This means that

$$|\langle \xi, Af \rangle_A + \langle \eta, Bg \rangle_B| \leq m_{\xi, \eta} \|Af + Bg\|.$$

On the other hand, since $\{\xi, \eta\} \in \text{dom } J^{**}$, there is a sequence $\{f_n, g_n\}_{n=1}^\infty$ from $\text{dom } A \times \text{dom } B$ such that $\{Af_n, Bg_n\} \rightarrow \{\xi, \eta\}$ and that $Af_n + Bg_n \rightarrow J^{**}\{\xi, \eta\}$. So, according to these considerations, a vector $y \in \mathfrak{H}$ can only be included by the range of $A_N \dot{+} B_N$ if there are sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ from $\text{dom } A$ and $\text{dom } B$, respectively, with $Af_n + Bg_n \rightarrow y$ such that $\{Af_n\}_{n=1}^\infty$ and $\{Bg_n\}_{n=1}^\infty$ are convergent in the corresponding associated Hilbert spaces and their limits satisfy

$$\lim_{n \rightarrow \infty} |\langle Af_n, Af \rangle_A + \langle Bg_n, Bg \rangle_B| = \lim_{n \rightarrow \infty} |(Af_n, f) + (Bg_n, g)| \leq m_y \|Af + Bg\|$$

for all $f \in \text{dom } A$ and $g \in \text{dom } B$ with some nonnegative constant m_y depending only on y . However, the sequences $\{Af_n\}_{n=1}^\infty$ and $\{Bg_n\}_{n=1}^\infty$ above are convergent if and only if they satisfy the Cauchy-property, i.e.

$$(A(f_n - f_m), f_n - f_m) \rightarrow 0 \quad \text{and} \quad (B(g_n - g_m), g_n - g_m) \rightarrow 0,$$

by letting $n, m \rightarrow \infty$. So, the equivalence of (i) and (ii) is has been proved. \square

Perhaps it is worth to examine the above characterizations in the spacial case when the original operators A and B are assumed to be selfadjoint:

Theorem 5.8. *Let A and B be positive selfadjoint operators in the Hilbert space \mathfrak{H} such that $\text{dom } A^{1/2} \cap \text{dom } B^{1/2}$ is dense. Then*

- (a) $\text{dom}(A \dot{+} B)^{1/2} = \text{dom } A^{1/2} \cap \text{dom } B^{1/2}$.
- (b) $\ker(A \dot{+} B)^{1/2} = \ker(A \dot{+} B) = \{\text{ran } A + \text{ran } B\}^\perp$.
- (c) *For any $y \in \mathfrak{H}$ the following statements are equivalent:*
 - (i) $y \in \text{ran}(A \dot{+} B)^{1/2}$;
 - (ii) *There is a nonnegative constant m_y such that for any $h \in \text{dom } A^{1/2} \cap \text{dom } B^{1/2}$*

$$|(y, h)|^2 \leq m_y (\|A^{1/2}h\|^2 + \|B^{1/2}h\|^2).$$
 - (iii) *There are two sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ from $\text{dom } A$ and $\text{dom } B$, respectively, such that $\{A^{1/2}f_n\}_{n=1}^\infty$ and $\{B^{1/2}g_n\}_{n=1}^\infty$ are convergent and that $Af_n + Bg_n \rightarrow y$.*
- (d) *For any $h \in \mathfrak{H}$ the following statements are equivalent*
 - (i) $h \in \text{dom}(A \dot{+} B)$;
 - (ii) $h \in \text{dom } A^{1/2} \cap \text{dom } B^{1/2}$ *and there are two sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ from $\text{dom } A$ and $\text{dom } B$, respectively, with $A^{1/2}f_n \rightarrow A^{1/2}h$, $B^{1/2}g_n \rightarrow B^{1/2}h$ such that $\{Af_n + Bg_n\}_{n=1}^\infty$ converges.*
- (e) *For any $y \in \mathfrak{H}$ the following statements are equivalent*
 - (i) $h \in \text{ran}(A \dot{+} B)$;
 - (ii) *There are two sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ from $\text{dom } A$ and $\text{dom } B$, respectively, such that $\{A^{1/2}f_n\}_{n=1}^\infty$ and $\{B^{1/2}g_n\}_{n=1}^\infty$ are convergent, $Af_n + Bg_n \rightarrow y$ and that for any $f \in \text{dom } A$ and $g \in \text{dom } B$*

$$\lim_{n \rightarrow \infty} |(A^{1/2}f_n, A^{1/2}f) + (B^{1/2}g_n, B^{1/2}g)| \leq m_y \|Af + Bg\|,$$

where m_y is a nonnegative constant depending only on y .

Proof. Parts (a) and (b) are immediate consequences of the Theorem 5.4 by only noticing that $\mathcal{D}_*(A) = \text{dom } A^{1/2}$ and $\mathcal{D}_*(B) = \text{dom } B^{1/2}$. Part (c) follows from Theorem 5.5 by observing that

$$M_A(h) = \langle J_A^* h, J_A^* h \rangle_A = \|A^{1/2}h\|^2,$$

and, analogously, that $M_B(h) = \|B^{1/2}h\|^2$. To prove (d), it suffices to show that statements (ii) (a) and (b) of Theorem 5.6 are equivalent to the assertions that $A^{1/2}f_n \rightarrow A^{1/2}h$ and $B^{1/2}g_n \rightarrow B^{1/2}h$. Nevertheless, each of these statements are equivalent to the following:

$$\langle J_A^*(f_n - h), J_A^*(f_n - h) \rangle_A \rightarrow 0 \quad \text{and} \quad \langle J_B^*(g_n - h), J_B^*(g_n - h) \rangle_B \rightarrow 0.$$

The details are left to the reader. Finally, statement (e) is again a direct consequence of the appropriate range characterization theorem above. \square

We close the section with a characterization of those positive operators whose form sum is bounded.

Corollary 5.9. *Let A and B be positive operators such that $\mathcal{D}_*(A) \cap \mathcal{D}_*(B)$ is dense. The following statements are equivalent:*

- (i) $A_N + B_N$ is bounded,
- (ii) $\mathcal{D}_*(A) = \mathcal{D}_*(B) = \mathfrak{H}$.

If either of the above assertions is fulfilled then both of A_N and B_N are bounded and $A_N + B_N = A_N + B_N$.

Proof. If $A_N + B_N = J^{**}J^*$ is bounded, then clearly

$$\mathcal{D}_*(A) \cap \mathcal{D}_*(B) = \text{dom } J^* \supseteq \text{dom } J^{**}J^* = \mathfrak{H},$$

thus (i) implies (ii). Conversely, in case (ii) both of the operators J_A^* and J_B^* are everywhere defined closed operators so that they are continuous according to the Banach' closed graph theorem. Thus both $A_N = J_A^{**}J_A^*$ and $B_N = J_B^{**}J_B^*$. Therefore $A_N + B_N$, being an everywhere defined bounded operator, does not admit any proper extensions. In particular, $J^{**}J^* = A_N + B_N$. \square

5.3. Form sums of positive operators with closed range

The first result of this section gives a characterization of those pairs of positive operators whose form sum has closed range. In [61] the author considered positive operators A whose Krein-von Neumann extension has closed range. By choosing $B = 0$ one easily obtains that $A_N + B_N = A_N$, which indicates that the situation discussed below is more general.

Theorem 5.10. *Let A and B be positive operators in the Hilbert space \mathfrak{H} such that $\mathcal{D}_*(A) \cap \mathcal{D}_*(B)$ is dense. The form sum of A_N and B_N has closed range if and only if there is constant $\gamma > 0$ such that*

$$(5.10) \quad M_A(h) + M_B(h) \geq \gamma \cdot \|h\|^2$$

for each $h \in \mathcal{D}_*(A) \cap \mathcal{D}_*(B) \cap \overline{\text{ran } A + \text{ran } B}$, where $M_A(h)$ and $M_B(h)$ are defined in (5.8). If this is the case then

$$\text{ran}(A_N + B_N) = \overline{\text{ran } A + \text{ran } B}.$$

Proof. According to Theorem 5.3, the form sum of A_N and B_N equals $J^{**}J^*$. Here, the range of $J^{**}J^*$ is closed if and only if $\text{ran } J^*$ is closed, see [61, Theorem 2.6]. However, J^* is a closed operator between the Hilbert spaces \mathfrak{H} and $\mathfrak{H}_A \times \mathfrak{H}_B$, so $\text{ran } J^*$ is closed if and only if J^* is bounded from below in the sense that there exists a positive constant $\gamma > 0$ such that

$$\|J^*h\|_{\mathfrak{H}_A \times \mathfrak{H}_B}^2 \geq \gamma \cdot \|h\|^2, \quad h \in \text{dom } J^* \cap \{\ker J^*\}^\perp.$$

On the other hand we have

$$\|J^*h\|_{\mathfrak{H}_A \times \mathfrak{H}_B}^2 = M_A(h) + M_B(h),$$

due to (5.9) which proves the first part of our statement.

Finally, if we assume that $\text{ran } J^{**}J^*$ is closed, then we have at the same time identity $\text{ran } J^{**} = \text{ran } J^{**}J^*$ which yields

$$\text{ran}(A_N + B_N) = \text{ran } J^{**} = \overline{\text{ran } J^{**}} = \overline{\text{ran } J} = \overline{\text{ran } A + \text{ran } B}.$$

□

Corollary 5.11. *Assume that A and B are positive selfadjoint operators such that $\text{dom } A^{1/2} \cap \text{dom } B^{1/2}$ is dense in \mathfrak{H} . Then $A + B$ has closed range if and only if there is a constant $\gamma > 0$ such that*

$$\|A^{1/2}h\|^2 + \|B^{1/2}h\|^2 \geq \gamma \cdot \|h\|^2$$

for all $h \in \text{dom } A^{1/2} \cap \text{dom } B^{1/2} \cap \overline{\text{ran } A + \text{ran } B}$.

Proof. It is a direct consequence of Theorem 5.10 by noticing that $\text{dom } A^{1/2} = \mathcal{D}_*(A)$, $\text{dom } B^{1/2} = \mathcal{D}_*(B)$ and that

$$M_A(h) = \langle J_A^*h, J_A^*h \rangle_A = \|A^{1/2}h\|^2, \quad M_B(h) = \langle J_B^*h, J_B^*h \rangle_B = \|B^{1/2}h\|^2$$

holds for all $h \in \text{dom } A^{1/2} \cap \text{dom } B^{1/2}$. □

If both of the operators A_N and B_N are bounded, i.e. when $\mathcal{D}_*(A) = \mathcal{D}_*(B) = \mathfrak{H}$, the situation becomes somewhat simpler (cf. also [60]):

Theorem 5.12. *Assume that A and B are positive operators in the Hilbert space \mathfrak{H} such that $\mathcal{D}_*(A) = \mathcal{D}_*(B) = \mathfrak{H}$. Then $A_N + B_N$ has closed range if and only if*

$$M_A(h) + M_B(h) \geq \gamma \cdot \|h\|^2$$

for any $h \in \text{ran } A + \text{ran } B$.

Proof. According to Corollary 5.9, $\text{dom } J^{**}J^* = \text{dom } J^* = \mathfrak{H}$ and $A_N \dot{+} B_N = A_N + B_N$ in this case. Thus condition (5.10) of Theorem 5.10 becomes

$$(5.11) \quad \|J^*h\|_{\mathfrak{H}_A \times \mathfrak{H}_B}^2 \geq \gamma \cdot \|h\|^2 \quad \text{for all } h \in \overline{\text{ran } A + \text{ran } B}.$$

Since J^* is continuous, it suffices to prescribe (5.11) only on $\text{ran } A + \text{ran } B$. \square

If T is a densely defined closed operator between the Hilbert spaces \mathfrak{H} and \mathfrak{K} such that $\text{ran } T$ is closed in \mathfrak{K} then there is a unique operator T^\dagger from \mathfrak{K} into \mathfrak{H} , called the *Moore-Penrose pseudoinverse* or *generalized inverse* of T , satisfying the following identities:

$$(5.12) \quad \begin{cases} \ker T^\dagger = \{\text{ran } T\}^\perp, \\ T^\dagger T h = P_{\{\ker T\}^\perp} h \quad \text{for all } h \in \text{dom } T, \\ T T^\dagger k = P_{\text{ran } T} k \quad \text{for all } k \in \mathfrak{K}. \end{cases}$$

For a given closed linear subspace \mathfrak{M} , its appropriate orthogonal projection is denoted by $P_{\mathfrak{M}}$. For further basic properties of Moore-Penrose inverses we refer to [8, 9, 10].

In the following result we construct the Moore-Penrose pseudoinverse of the form sum of two positive operators.

Theorem 5.13. *Let A and B be positive operators in the Hilbert space \mathfrak{H} such that $\mathcal{D}_*(A) \cap \mathcal{D}_*(B)$ is dense and that $A_N \dot{+} B_N$ has closed range. Then we have*

$$(5.13) \quad \text{ran } A + \text{ran } B \subseteq \text{ran}(A_N \dot{+} B_N),$$

i.e. for any $f \in \text{dom } A$ and $g \in \text{dom } B$ there is a $h \in \text{dom } J^{**}J^*$ with $J^{**}J^*h = Af + Bg$. If V denotes the mapping from \mathfrak{H} into $\mathfrak{H}_A \times \mathfrak{H}_B$ defined by the following correspondence

$$(5.14) \quad \begin{aligned} \text{dom } V &= \{Af + Bg + k : f \in \text{dom } A, g \in \text{dom } B, k \in \{\text{ran } A + \text{ran } B\}^\perp\}, \\ Af + Bg + k &\mapsto J^*h, \end{aligned}$$

then V is a densely defined bounded linear operator such that

$$(5.15) \quad V^{**}V^* = (A_N \dot{+} B_N)^\dagger,$$

that is $V^{**}V^*$ equals the Moore-Penrose pseudoinverse of $A_N \dot{+} B_N$. Furthermore, the norm $\|(A_N \dot{+} B_N)^\dagger\|$ can be computed as the following infimum:

$$\inf \{ \gamma \geq 0 : \|k\|^2 \leq \gamma(M_A(k) + M_B(k)) \text{ for all } k \in \mathcal{D}_*(A) \cap \mathcal{D}_*(B) \cap \overline{\text{ran } A + \text{ran } B} \},$$

where $M_A(k)$ and $M_B(k)$ are defined in (5.8).

Proof. Since $J^{**}J^*$ has closed range by assumption, (5.13) is obtained as follows:

$$\text{ran } A + \text{ran } B = \text{ran } J \subseteq \text{ran } J^{**} = \text{ran } J^{**}J^*.$$

(We mention that equality $\text{ran } J^{**} = \text{ran } J^{**}J^*$ holds if and only if $J^{**}J^*$ has closed range, cf. [61]; see also [14, 60] for the case of bounded operators.) On the other hand, if we put vectors $f \in \text{dom } A$ and $g \in \text{dom } B$ and $h, h' \in \text{dom } J^{**}J^*$ such that

$$J^{**}J^*h = Af + Bg = J^{**}J^*h',$$

then $J^*h - J^*h' \in \text{ran } J^* \cap \{\ker J^{**}\}^\perp = 0$. Therefore the correspondence (5.14) gives a well defined linear operator, indeed. Firstly we show that V is continuous: since J^{**} has closed range, it is bounded from below by a positive constant $\gamma > 0$ so that for any $Af + Bg + k \in \text{dom } V$ we have

$$\begin{aligned} \|V(Af + Bg + k)\|^2 &= \|J^*h\|^2 \leq \gamma \cdot \|J^{**}J^*h\|^2 = \gamma \cdot \|Af + Bg\|^2 \\ &\leq \gamma \cdot \|Af + Bg + k\|^2. \end{aligned}$$

Therefore, V is continuous by norm bound $\sqrt{\gamma}$ so that it admits its unique norm preserving continuous extension to \mathfrak{H} , namely V^{**} , with the same bound.

In order to show identity (5.15) we are going to prove first that $V^{**} = (J^{**})^\dagger$, that is V^{**} and J^{**} fulfill identities in (5.12). It is clear that

$$\ker V^{**} \supseteq \ker V = \{\text{ran } A + \text{ran } B\}^\perp = \{\text{ran } J^{**}\}^\perp.$$

To the reverse inclusion, let $v \in \ker V^{**}$ and put a sequence $\{Af_n + Bg_n + k_n\}_{n=1}^\infty$ from $\text{dom } V$ such that

$$Af_n + Bg_n + k_n \rightarrow v \quad \text{and} \quad V(Af_n + Bg_n + k_n) \rightarrow 0.$$

Since $J^{**}V(Af_n + Bg_n + k_n) = Af_n + Bg_n$ for each $n \in \mathbb{N}$, it follows that

$$Af_n + Bg_n \rightarrow 0 \quad (n \rightarrow \infty),$$

according to the closedness of J^{**} . Thus we obtain that $k_n \rightarrow v$ which yields $v \in \{\text{ran } J^{**}\}^\perp$. To get the second equality of (5.12) let $\xi \in \text{dom } J^{**}$. Since the range space of a densely defined closed operator is closed if and only if the range of its adjoint is closed (see [27] or [61]), it follows that $\text{ran } J^*$ is closed too. Therefore, ξ has the orthogonal decomposition $J^*h + \eta$ with respect to $\text{ran } J^* \oplus \ker J^{**}$ so that $h \in \text{dom } J^{**}J^*$ and that $J^{**}\xi = J^{**}J^*h$. We should prove identity

$$(5.16) \quad V^{**}J^{**}\xi = J^*h.$$

Since J^{**} is just the closure of J , there is a sequence $\{Af_n, Bg_n\}_{n=1}^\infty$ from $\text{dom } J$ such that

$$\{Af_n, Bg_n\} \rightarrow \xi \quad \text{and} \quad Af_n + Bg_n \rightarrow J^{**}\xi = J^{**}J^*h.$$

On the other hand for each $n \in \mathbb{N}$ we have $V(Af_n + Bg_n) \in \text{dom } J^{**}$ so that

$$V(Af_n + Bg_n) \rightarrow V^{**}J^{**}\xi \quad \text{and} \quad J^{**}V(Af_n + Bg_n) = Af_n + Bg_n \rightarrow J^{**}J^*h,$$

which implies $V^{**}J^{**}\xi \in \text{dom } J^{**}$ and that

$$J^{**}V^{**}J^{**}\xi = J^{**}J^*h$$

according to the closedness of J^{**} . This follows equality (5.16), as desired. Finally, to prove the third equation of (5.12) pick a $h \in \mathfrak{H}$. We should prove that $V^{**}h \in \text{dom } J^{**}$

and that

$$(5.17) \quad J^{**}V^{**}h = Ph,$$

where P stands for the orthogonal projection of \mathfrak{H} onto $\text{ran } J^{**}$. Let $\{Af_n + Bg_n + k_n\}_{n=1}^{\infty}$ be a sequence from $\text{dom } V$ that tends to h . Since for each $n \in \mathbb{N}$ we have $V(Af_n + Bg_n) \in \text{dom } J^{**}$ so that

$$V(Af_n + Bg_n) \rightarrow V^{**}h \quad \text{and} \quad J^{**}V(Af_n + Bg_n) = Af_n + Bg_n \rightarrow Ph,$$

according again to the closedness of J^{**} , we obtain that $V^{**}h \in \text{dom } J^{**}$ and that (5.17) holds. Therefore, $V^{**} = (J^{**})^\dagger$, as desired. At the same time, $V^* = (J^*)^\dagger$ according to the basic properties of the Moore-Penrose pseudoinverse. And now we have arrived: V^*V^{**} and $J^{**}J^*$ satisfy the identities of (5.12). Indeed, by denoting the orthogonal projections onto $\text{ran } J^{**}$ and $\text{ran } J^*$ by P and Q , respectively, we conclude that

$$\begin{cases} \ker V^*V^{**} = \ker V^{**} = \{\text{ran } J^{**}\}^\perp = \{\text{ran } J^{**}J^*\}^\perp, \\ V^*V^{**}J^{**}J^*h = V^*QJ^*h = V^*J^*h = Ph & \text{for all } h \in \text{dom } J^{**}J^*, \\ J^{**}J^*V^*V^{**}h = J^{**}QV^{**}h = J^{**}V^{**}h = Ph & \text{for all } h \in \mathfrak{H}, \end{cases}$$

and this yields just identity (5.15).

It only remains to prove the formula concerning the norm. According to the basic properties of the Moore-Penrose pseudoinverse we have the following equalities:

$$\begin{aligned} \|V^*V^{**}\| &= \|V^*\|^2 = \|(J^*)^\dagger\|^2 \\ &= \inf\{\gamma \geq 0 : \|k\|^2 \leq \gamma \cdot \|J^*k\|^2 \text{ for all } k \in \text{dom } J^* \cap \text{ran } J^{**}\}. \end{aligned}$$

The following identities

$$\|J^*k\|^2 = M_A(k) + M_B(k), \text{dom } J^* = \mathcal{D}_*(A) \cap \mathcal{D}_*(B), \text{ran } J^{**} = \overline{\text{ran } A + \text{ran } B}$$

yield the desired formula. □

CHAPTER 6

Biorthogonal expansions for symmetrizable operators

Throughout this chapter we assume that a positive linear operator A is given on a complex Hilbert space \mathfrak{H} with inner product (\cdot, \cdot) . Positive operator is in other words nonnegative, that is the quadratic form of which is nonnegative. Another bounded linear operator B is also given on \mathfrak{H} with the extra property that the product operator AB is selfadjoint, that is the identity

$$AB = B^*A$$

is satisfied. This fact is also expressed by saying that B is *symmetrizable* with respect to A (on the left).

The question of symmetrizability of operators goes back to Krein [34], Zaanen [65], Reid [41], Dieudonné [12] and Lax [35]. An essential restriction is generally given by the nondegeneracy of the positive operator A . We make no use of this restriction.

Our approach is based on the construction of [46], which has been already expounded in the introduction of Chapter 5. However, thanks to the continuity of the positive operator A , the natural embedding J_A of $\text{ran } A$ into \mathfrak{H} is now bounded by norm $\|A\|^{1/2}$, in virtue of the operator Schwarz inequality

$$\|Ax\|^2 \leq \|A\|(Ax, x) \quad (x \in \mathfrak{H}).$$

The unique continuous extension J_A^{**} of J_A is a one-one continuous map of \mathfrak{H}_A onto $\text{ran } A^{1/2}$ in \mathfrak{H} having the adjoint

$$J_A^*x = Ax \in \mathfrak{H}_A \quad (x \in \mathfrak{H});$$

hence the factorization

$$J_A^{**}J_A^* = A$$

for our original operator holds. On the other hand, Theorem 1 of [47] gives also factorization for the selfadjoint product AB with a symmetrizable operator B in the form

$$J_A^{**}\widehat{B}J_A^* = AB,$$

where \widehat{B} is a selfadjoint operator on the constructed Hilbert space \mathfrak{H}_A such that

$$\widehat{B}(Ax) = ABx \quad (x \in \mathfrak{H}).$$

The inclusion

$$(6.1) \quad \mathrm{Sp}(\widehat{B}) \subseteq \mathrm{Sp}(B)$$

holds with respect to the spectra (hence the norm estimation $\|\widehat{B}\| \leq r(B)$ holds also satisfied where $r(B)$ denotes the spectral radius of B).

It turns out that \mathfrak{H}_A is mapped onto $\overline{\mathrm{ran} A}$ by the isometry U given on (a dense manifold) $\mathrm{ran} A$ as follows (see [47, Theorem 3]):

$$(6.2) \quad U(Ax) := A^{1/2}x \in \mathfrak{H} \quad (x \in \mathfrak{H}),$$

therefore

$$(6.3) \quad U^*(A^{1/2}x) := Ax \in \mathfrak{H}_A \quad (x \in \mathfrak{H}).$$

As a consequence we have the selfadjoint operator S on \mathfrak{H} :

$$(6.4) \quad S := U\widehat{B}U^*,$$

which satisfies the characteristic property of Dieudonné [12]

$$(6.5) \quad SA^{1/2} = A^{1/2}B.$$

6.1. Characterization of symmetrizable operators

We give in what follows some characterization for the symmetrizable operator B by using the model space \mathfrak{H}_A . These problems first appeared in the classical paper by Krein [34].

The symmetrizable operator in the next lemma is called H - B continuous by Krein.

Lemma 6.1. *Let A, B be bounded operators on a Hilbert space \mathfrak{H} such that A is positive. The following assertions are equivalent:*

- (i) *The set $\{Bx : x \in \mathfrak{H}, (Ax, x) \leq 1\}$ is bounded in \mathfrak{H} ;*
- (ii) *There exists a bounded linear operator V from \mathfrak{H}_A into \mathfrak{H} satisfying the operator equality*

$$VJ_A^* = B$$

or in other words, $V(Ax) = Bx$ for all x from \mathfrak{H} ;

- (iii) *The range inclusion $\mathrm{ran} B^* \subseteq \mathrm{ran} A^{1/2}$ holds true.*

Proof. Assuming (i), the mapping $V : \mathfrak{H}_A \supseteq \mathrm{ran} A \rightarrow \mathfrak{H}$ defined by the following relation

$$Ax \mapsto Bx \quad (x \in \mathfrak{H})$$

gives a bounded linear operator from \mathfrak{H}_A into \mathfrak{H} with the dense domain $\mathrm{ran} A$ so that its unique continuous extension satisfies (ii). If we assume (ii) then we know the factorization

of B as follows

$$B = VJ_A^*.$$

Hence $B^* = J_A^{**}V^*$ and the statement in (iii) follows:

$$\text{ran } B^* \subseteq \text{ran } J_A^{**} = \text{ran}(J_A^* J_A^*)^{1/2} = \text{ran } A^{1/2}.$$

At last assume (iii) and prove (i): this is just the Douglas' factorization theorem in [15]. The proof is complete. \square

Note that condition (ii) in the above lemma implies the identity

$$\widehat{B} = J_A^* V$$

for a symmetrizable operator B :

$$J_A^* V(Ax) = J_A^*(Bx) = ABx = \widehat{B}(Ax) \quad (x \in \mathfrak{H}).$$

The next characterization of a symmetrizable operator B was called *H-compact* by Krein.

Proposition 6.2. *Let B be a symmetrizable operator on a Hilbert space \mathfrak{H} with respect to a positive operator A . The following assertions are equivalent:*

- (a) *The set $\{A^{1/2}Bx : x \in \mathfrak{H}, (Ax, x) \leq 1\}$ is precompact in \mathfrak{H} ;*
- (b) *There exists a compact operator W from \mathfrak{H}_A into \mathfrak{H} satisfying the operator equality*

$$(6.6) \quad WJ_A^* = A^{1/2}B,$$

or in other words $W(Ax) = A^{1/2}Bx$ for all x from \mathfrak{H} ;

- (c) *\widehat{B} is compact operator on \mathfrak{H}_A ;*
- (d) *S is compact operator on \mathfrak{H} .*

Proof. Clearly, (a) implies (b). Assuming (b) we have the factorization

$$U^*W = \widehat{B}$$

where U is the isometry defined in (6.2):

$$U^*W(Ax) = U^*(A^{1/2}Bx) = ABx = \widehat{B}(Ax) \quad (x \in \mathfrak{H})$$

and hence the compactness of \widehat{B} is obvious. From the definition of S in (6.4) we have that (c) implies (d), and if S is compact then using identities (6.5) and (6.3) the set

$$\begin{aligned} \{A^{1/2}Bx : x \in \mathfrak{H}, (Ax, x) \leq 1\} &= \{SA^{1/2}x : x \in \mathfrak{H}, (Ax, x) \leq 1\} \\ &= \{SU(Ax) : x \in \mathfrak{H}, \langle Ax, Ax \rangle_A \leq 1\} \end{aligned}$$

is precompact, i.e. (d) implies the statement in (a). \square

Theorem 6.3. *Let B be a symmetrizable operator on a Hilbert space \mathfrak{H} with respect to a positive operator A . If in addition B is assumed to be compact then each of (a)-(d) are satisfied.*

Proof. The Riesz-Schauder theory of compact operators says that $\text{Sp}(B)$ may have only one point of accumulation namely the point 0. According to the spectra inclusion in (6.1), the same holds for the selfadjoint operator \widehat{B} . As a consequence any nonzero spectrum point of \widehat{B} is an eigenvalue. The compactness of \widehat{B} follows then by showing that any nonzero eigenvalue λ is of finite multiplicity, i.e. the kernel of $\widehat{B} - \lambda\widehat{I}$ is finite dimensional. We are going to show this step by step.

On one hand $\ker(\widehat{B} - \lambda\widehat{I}) \cap \text{ran } A$ consists of

$$\begin{aligned} \{Ay : y \in \overline{\text{ran } A}, (\widehat{B} - \lambda\widehat{I})Ay = 0\} &= \{Ay : y \in \overline{\text{ran } A}, A(B - \lambda I)y = 0\} \\ &= \{Ay : y \in \overline{\text{ran } A}, (B - \lambda I)y \in \ker A\} \\ &= \{Ay : y \in \overline{\text{ran } A} \cap \ker(PB - \lambda I)\}, \end{aligned}$$

where P is the orthogonal projection of \mathfrak{H} onto $\overline{\text{ran } A}$ and PB being compact $\ker(PB - \lambda I)$ is finite dimensional on the other hand. Let y_1, \dots, y_n be a maximal linear independent family of such vectors and

$$(6.7) \quad \mathfrak{H}^\lambda := \{x \in \overline{\text{ran } A} : (x, Ay_j) = 0, \quad (j = 1, 2, \dots, n)\}.$$

Then PB leaves \mathfrak{H}^λ invariant: for each x from \mathfrak{H}^λ and y_j , $j = 1, 2, \dots, n$ we have that

$$\begin{aligned} (PBx, Ay_j) &= (Bx, Ay_j) = (x, AB y_j) = (x, APB y_j) = (x, A\lambda y_j) \\ &= \overline{\lambda}(x, Ay_j) = 0. \end{aligned}$$

Hence $B_\lambda := PB|_{\mathfrak{H}^\lambda}$ is compact operator on the Hilbert space \mathfrak{H}^λ such that λ is not an eigenvalue for it: suppose that $PBx = \lambda x$ for some $x \in \mathfrak{H}^\lambda$ then it follows that $x \in \ker(PB - \lambda I) \cap \overline{\text{ran } A}$, i.e.

$$x = \sum_{k=1}^n \mu_k y_k$$

holds for some μ_1, \dots, μ_n scalars. Thus

$$(Ax, x) = \sum_{k=1}^n (Ax, \mu_k y_k) = \sum_{k=1}^n \overline{\mu_k} (x, Ay_k) = 0,$$

hence $Ax = 0$, and $x = 0$ because $x \in \overline{\text{ran } A}$. In other words λ is not a spectrum point of B_λ . But then the operator compression $A_\lambda := P_\lambda A|_{\mathfrak{H}^\lambda}$, where P_λ is the orthogonal projection of \mathfrak{H} onto \mathfrak{H}^λ , is a positive operator on \mathfrak{H}^λ such that B_λ is symmetrizable with respect to it:

$$\begin{aligned} (A_\lambda B_\lambda x, y) &= (P_\lambda APBx, y) = (ABx, y) = (x, AB y) = (P_\lambda x, APB y) \\ &= (x, A_\lambda B_\lambda y) \end{aligned}$$

for $x, y \in \mathfrak{H}^\lambda$, i.e. $(A_\lambda B_\lambda)^* = A_\lambda B_\lambda$ indeed. The corresponding selfadjoint operator \widehat{B}_λ on the so auxiliary Hilbert space \mathfrak{H}_A^λ has not λ as spectrum point. Here we have arrived: \widehat{B} acts on the orthogonal complement L of $\ker(\widehat{B} - \lambda I) \cap \text{ran } A$ just as a unitary equivalent of \widehat{B}_λ . To see this note that $L \cap \text{ran } A$ is dense in L since L is of finite codimension, cf. [28]. This means that

$$\begin{aligned} L \cap \text{ran } A &= \{Ax : x \in \overline{\text{ran } A}, \langle Ax, Ay_j \rangle_A = 0 \quad (j = 1, 2, \dots, n)\} \\ &= \{Ax : x \in \overline{\text{ran } A}, (x, Ay_j) = 0 \quad (j = 1, 2, \dots, n)\} \\ &= \{Ax : x \in \mathfrak{H}^\lambda\} \end{aligned}$$

is dense in L . Hence the map given on $L \cap \text{ran } A$ as follows

$$\Psi(Ax) := A_\lambda x \in \mathfrak{H}_A^\lambda \quad (x \in \mathfrak{H}^\lambda)$$

is a densely defined isometry with the dense range $\text{ran } A_\lambda$ and its unique continuous extension, also denoted by Ψ , defines a unitary operator between the Hilbert spaces L and \mathfrak{H}_A^λ . Furthermore $\widehat{B}|_L$ and \widehat{B}_λ are unitary equivalent under this unitary map:

$$\Psi^{-1} \widehat{B}_\lambda \Psi(Ax) = \Psi^{-1}(A_\lambda B_\lambda x) = APBx = ABx = \widehat{B}(Ax) \quad (x \in \mathfrak{H}^\lambda).$$

Consequently, the fact that λ is not a spectrum point of \widehat{B}_λ implies that $\ker(\widehat{B}|_L - \lambda I) = \{0\}$ and hence $\ker(\widehat{B} - \lambda I)$ is of finite dimension. The proof is complete. \square

6.2. Biorthogonal expansions for symmetrizable operators

Given a symmetrizable operator B satisfying the assumptions made in Lemma 6.1 and Proposition 6.2 we see that S is a compact selfadjoint operator so that the factorization

$$S = U \widehat{B} U^* = U J_A^* V U^*$$

holds and hence the range inclusion

$$\text{ran } S \subseteq \text{ran } U J_A^* = \text{ran } A^{1/2}$$

follows. The Hilbert-Schmidt theory says that there exists an orthonormal sequence $\{A^{1/2}e_n\}_{n \in \mathbb{N}}$ with $e_n \in \overline{\text{ran } A}$ such that $A^{1/2}e_n$ is the eigenvector for the corresponding nonzero real eigenvalue λ_n of S . If S is not of finite rank it can be assumed that $\{|\lambda_n|\}_{n \in \mathbb{N}}$ is strictly decreasing sequence with the limit point 0. Moreover the spectral expansion

$$(6.8) \quad Sx = \sum_{k=1}^{\infty} \lambda_k(x, A^{1/2}e_k) A^{1/2}e_k$$

holds also true and

$$(6.9) \quad S_n x = \sum_{k=1}^n \lambda_k(x, A^{1/2}e_k) A^{1/2}e_k$$

forms a finite rank selfadjoint operator sequence with norm limit S . Also observe that

$$\|S_n x\| \leq \|Sx\| \quad (n \in \mathbb{N}).$$

Note also that $\{J_A^* e_n\}_{n \in \mathbb{N}}$ is an orthonormal sequence in the constructed Hilbert space \mathfrak{H}_A .

Now we are ready to prove the result indicated in the title of this note.

Theorem 6.4. *Let B be a symmetrizable operator satisfying any of the conditions of Lemma 6.1 and Proposition 6.2. Then there exists a biorthogonal sequence of vectors $\{e_n, f_n\}_{n \in \mathbb{N}}$ such that the following spectral expansions hold*

$$(6.10) \quad PBx = \sum_{k=1}^{\infty} \lambda_k(x, f_k) e_k \quad (x \in \mathfrak{H}),$$

$$(6.11) \quad B^*Px = \sum_{k=1}^{\infty} \lambda_k(x, e_k) f_k \quad (x \in \mathfrak{H}),$$

where the convergence is in the norm of the Hilbert space and P stands for the orthogonal projection of \mathfrak{H} onto $\overline{\text{ran } A}$.

Proof. Let $\{e_n\}_{n \in \mathbb{N}}$ be the sequence above and define

$$f_n := Ae_n \quad (n \in \mathbb{N})$$

so that $\{e_k\}_{k \in \mathbb{N}}$ and $\{f_n\}_{n \in \mathbb{N}}$ have the biorthogonal property

$$(f_n, e_k) = (Ae_n, e_k) = (A^{1/2}e_n, A^{1/2}e_k) = \delta_n^k \quad (n, k \in \mathbb{N}).$$

Note now that

$$A^{1/2}PB e_k = A^{1/2}B e_k = S A^{1/2}e_k = \lambda_k A^{1/2}e_k \quad (k \in \mathbb{N})$$

means that

$$PB e_k = \lambda_k e_k \quad (k \in \mathbb{N}).$$

Therefore for the partial sums

$$T_n x := \sum_{k=1}^n \lambda_k(x, f_k) e_k = \sum_{k=1}^n (x, Ae_k) PB e_k \quad (n \in \mathbb{N})$$

we find the Cauchy property as follows:

$$\begin{aligned} \|T_n x - T_m x\|^2 &= \left\| \sum_{k=m+1}^n (x, Ae_k) PB e_k \right\|^2 = \left\| P \sum_{k=m+1}^n (x, Ae_k) V J_A^* e_k \right\|^2 \\ &\leq \|PV\|^2 \left\| \sum_{k=m+1}^n (A^{1/2}x, A^{1/2}e_k) J_A^* e_k \right\|_A^2 \\ &= \|PV\|^2 \sum_{k=m+1}^n |(A^{1/2}x, A^{1/2}e_k)|^2 \rightarrow 0, \quad (n, m \rightarrow \infty). \end{aligned}$$

Here we have used the appropriate factorization $B = VJ_A^*$ and the fact that both $\{J_A^*e_k\}_{k \in \mathbb{N}}$ and $\{A^{1/2}e_k\}_{k \in \mathbb{N}}$ are orthonormal sequences in the corresponding Hilbert spaces. Moreover we have used that $A^{1/2}x$ in \mathfrak{H} has Fourier coefficients $\{(A^{1/2}x, A^{1/2}e_k)\}_{k \in \mathbb{N}}$ with respect to the last orthonormal sequence. Observing now that

$$A^{1/2}T_n = S_n A^{1/2} \rightarrow S A^{1/2} \quad (n \rightarrow \infty)$$

in operator norm, we have that the limit satisfies

$$A^{1/2} \sum_{k=1}^{\infty} \lambda_k(x, f_k) e_k = S A^{1/2} x = A^{1/2} P B x \quad (x \in \mathfrak{H}).$$

Consequently, by noticing that $P e_k = e_k$ for all $k \in \mathbb{N}$ we obtain that

$$\sum_{k=1}^{\infty} \lambda_k(x, f_k) e_k - P B x \in \ker A^{1/2} \cap \overline{\text{ran } A}$$

for all $x \in \mathfrak{H}$ which implies (6.10).

We have on the other hand for all $x \in \mathfrak{H}$

$$T_n^* x = \sum_{k=1}^n \lambda_k(x, e_k) f_k$$

and hence that

$$T_n^* A^{1/2} = A^{1/2} S_n \rightarrow A^{1/2} S \quad (n \rightarrow \infty)$$

in operator norm. We have to check, of course, that $\{T_n^* x\}_{n \in \mathbb{N}}$ converges as follows

$$\begin{aligned} \|T_n^* x - T_m^* x\|^2 &= \left\| \sum_{k=m+1}^n \lambda_k(x, e_k) A e_k \right\|^2 = \left\| \sum_{k=m+1}^n (x, P B e_k) J_A^{**} J_A^* e_k \right\|^2 \\ &\leq \|J_A^{**}\|^2 \left\| \sum_{k=m+1}^n \langle V^* P x, J_A^* e_k \rangle_A J_A^* e_k \right\|_A^2 \\ &= \|A\| \sum_{k=m+1}^n |\langle V^* P x, J_A^* e_k \rangle_A|^2 \rightarrow 0 \quad (n, m \rightarrow \infty). \end{aligned}$$

Here the same argument applies, but reverse order: $\{A^{1/2}e_k\}_{k \in \mathbb{N}}$ forms orthonormal set in \mathfrak{H} and $V^* P x$ has Fourier coefficients $\{\langle V^* P x, J_A^* e_k \rangle_A\}_{k \in \mathbb{N}}$ with respect to the orthonormal set $\{J_A^* e_k\}_{k \in \mathbb{N}}$ in \mathfrak{H}_A . As a consequence we see that

$$(6.12) \quad \sum_{k=1}^{\infty} \lambda_k(A^{1/2}x, e_k) f_k = A^{1/2} S x = B^* A^{1/2} x = B^* P A^{1/2} x \quad (x \in \mathfrak{H})$$

We have at the same time uniform boundedness for $\{T_n^*\}_{n \in \mathbb{N}}$ in norm as follows

$$\|T_n^* x\|^2 \leq \|A\| \sum_{k=1}^n |\langle V^* P x, J_A^* e_k \rangle_A|^2 \leq \|A\| \|V^* P x\|_A^2 \leq \|A\| \|V^* P\|^2 \|x\|^2$$

for all $x \in \mathfrak{H}$. This together with (6.12) implies the expansion in (6.11) by noticing that $\text{ran } A^{1/2}$ is dense in $\overline{\text{ran } A}$. The proof is complete. \square

Corollary 6.5. *Let B be compact symmetrizable operator such that the range inclusion $\text{ran } B^* \subseteq \text{ran } A^{1/2}$ is satisfied. Then the biorthogonal expansions (6.10) and (6.11) hold true.*

We close with an extension of Theorem 9 by Krein [34].

Theorem 6.6. *Let T be a selfadjoint operator such that $TA^{1/2}$ is compact. Then there exists an A -orthonormal sequence $\{e_k\}_{k \in \mathbb{N}}$ in the sense $(Ae_k, e_l) = \delta_l^k$ for $k, l \in \mathbb{N}$ and a sequence of real numbers $\{\lambda_k\}_{k \in \mathbb{N}}$ which converges to zero such that the spectral expansion satisfies*

$$(6.13) \quad PTPx = \sum_{k=1}^{\infty} \lambda_k(x, e_k)e_k \quad (x \in \mathfrak{H})$$

where P is the orthonormal projection onto $\overline{\text{ran } A}$ and the convergence for PTP is uniform in norm on the unit ball.

Proof. Take $B = TA$ and Theorem 6.4 applies as follows. Denote by K_n the n -th partial sum in (6.13). Then as in the proof of Theorem 6.4

$$\begin{aligned} \|A^{1/2}K_mx - A^{1/2}K_nx\|^2 &= \left\| \sum_{k=m+1}^n \lambda_k(x, e_k)A^{1/2}e_k \right\|^2 = \sum_{k=m+1}^n |\lambda_k(x, e_k)|^2 \\ &= \sum_{k=m+1}^n |\langle V^*Px, J_A^*e_k \rangle|^2 \rightarrow 0, \quad (n, m \rightarrow \infty) \end{aligned}$$

according to the Fourier-expansion of V^*Px with respect to the orthonormal sequence $\{J_A^*e_k\}_{k \in \mathbb{N}}$. At the same time

$$\|A^{1/2}K_nx\|^2 \leq \|V^*Px\|_A^2 \quad (x \in \mathfrak{H}).$$

Therefore for each x from \mathfrak{H} we find that

$$\begin{aligned} A^{1/2} \sum_{k=1}^{\infty} \lambda_k(A^{1/2}x, e_k)e_k &= \sum_{k=1}^{\infty} \lambda_k(x, A^{1/2}e_k)A^{1/2}e_k = Sx, \\ A^{1/2} \sum_{k=1}^{\infty} \lambda_k(Ax, e_k)e_k &= SA^{1/2}x = A^{1/2}Bx = A^{1/2}TAx, \end{aligned}$$

and hence that

$$A^{1/2} \sum_{k=1}^{\infty} \lambda_k(x, e_k)e_k = A^{1/2}T(Px)$$

uniformly on the unit ball and finally just identity (6.13). The uniform convergence on the unit ball will be proved if we show the compactness of V^* . To see this observe first that

$$VJ_A^*x = Bx = TAx = TJ_A^{**}J_A^*x \quad (x \in \mathfrak{H})$$

leads us to the equality $V = TJ_A^{**}$ according to the density of $\text{ran } J_A^*$ in \mathfrak{H}_A . Since $TA^{1/2}$ is compact by assumption the following identities insure compactness for VV^* and consequently for V^* :

$$TAT = TJ_A^{**}J_A^*T = VV^*.$$

Note that the starting assumptions of Theorem 6.4 on B and A hold since

$$\text{ran } B^* = \text{ran } AT \subseteq \text{ran } A \subseteq \text{ran } A^{1/2}$$

and

$$\{A^{1/2}Bx : x \in \mathfrak{H}, (Ax, x) \leq 1\} = \{A^{1/2}TA^{1/2}(A^{1/2}x) : x \in \mathfrak{H}, \|A^{1/2}x\|^2 \leq 1\}$$

is precompact by our assumption. Of course, if we assume compactness on T as Krein did (together with positivity of T) our statement also remains true. \square

Lebesgue-type decomposition of positive operators

Throughout this chapter two bounded positive linear operators A and B are given on a complex Hilbert space \mathfrak{H} ; in notation $A, B \in \mathcal{B}(\mathfrak{H})$. Positivity means here that the quadratic forms of the corresponding operators are nonnegative semidefinite. The set of all bounded linear operators between two Hilbert spaces \mathfrak{H} and \mathfrak{K} will be denoted by $\mathcal{B}(\mathfrak{H}, \mathfrak{K})$. The positive operator B is called *absolutely continuous* (or in other words *closable*, cf. Simon [57]) with respect to A if for any sequence $\{x_n\}_{n=1}^{\infty}$ from \mathfrak{H} with $(Ax_n, x_n) \rightarrow 0$ and $(B(x_n - x_m), x_n - x_m) \rightarrow 0$ yields $(Bx_n, x_n) \rightarrow 0$. Furthermore, B is *singular* with respect to A if for any positive linear operator C the properties $C \leq A$ and $C \leq B$ hold only for $C = 0$. We will also refer to these concepts as A -absolute continuity and A -singularity as well.

These notions go back to Anderson and Trapp [1], see also Ando [2]. We notice that the original definition of A -absolute continuity introduced in [2] reads as follows: there are nondecreasing sequences of positive operators $\{B_n\}_{n=1}^{\infty}$ and of nonnegative numbers $\{\alpha_n\}_{n=1}^{\infty}$ such that

$$B_n \leq \alpha_n A \quad \text{and} \quad Bx = \lim_{n \rightarrow \infty} B_n x$$

holds for all $n \in \mathbb{N}$ and $x \in \mathfrak{H}$. This second definition of A -absolute continuity is equivalent to the closability above in view of Theorem 7.7 below (see also [25, Theorem 3.8]).

The sum $B = B_c + B_s$ is called Lebesgue-type decomposition of B with respect to A if B_c and B_s are both positive linear operators such that B_c is A -absolute continuous and B_s is A -singular, respectively.

Our primary reference is [46] of Z. Sebestyén. His construction described in the introduction of Chapter 5 for giving the Krein–von Neumann extension of a subpositive operator forms the basis of our treatment. Our secondary basic reference is [24] of Hassi et al. for canonical decompositions of linear relations.

Observe that the natural embedding operator J_A from the dense linear manifold $\text{ran } A$ of the auxiliary Hilbert space \mathfrak{H}_A into \mathfrak{H} is now continuous thanks to the operator Schwarz inequality

$$(Ax, Ax) \leq \|A\|(Ax, x), \quad (x \in \mathfrak{H}),$$

and that its unique continuous extension $J_A^{**} \in \mathcal{B}(\mathfrak{H}_A, \mathfrak{H})$ is one-to-one correspondence, having the range

$$\text{ran } J_A^{**} = \text{ran}(J_A^{**} J_A^*)^{1/2} = \text{ran } A^{1/2}.$$

7.1. Lebesgue type decomposition for positive operators

The method of Ando [2] for giving the absolutely continuous part of a positive operator with respect to another is based on the notion of parallel sums of positive operators given by the following formula with respect to their quadratic forms:

$$((A : B)x, x) := \inf_{y \in \mathfrak{H}} \{ (Ay, y) + (B(x - y), x - y) \}, \quad x \in \mathfrak{H}.$$

For the corresponding notion for nonnegative forms see [25]. It is proved in [2] that

$$(7.1) \quad [A]Bx := \lim_{n \rightarrow \infty} (A : nB)x, \quad x \in \mathfrak{H},$$

defines an A -absolute continuous bounded positive linear operator such that $B - [A]B$ is at the same time A -singular. Moreover, the following Lebesgue-type decomposition

$$B = [A]B + (B - [A]B)$$

turns out to be extremal in the sense that $[A]B$ is the maximum of all A -absolute continuous positive linear operators C with $C \leq B$.

As a matter of fact, Ando's approach considers bounded positive linear operators just as semi-inner products by identifying them with their quadratic forms via the Riesz representation theorem. That gave a rise to generalizing his method for the case of nonnegative forms, cf. [25] and [56]. Our treatment below is purely operator theoretic which uses factorizations via the auxiliary Hilbert spaces \mathfrak{H}_A and \mathfrak{H}_B associated to the positive operators A and B , respectively. It turns out that the absolute continuity, the singularity and also the uniqueness of the Lebesgue decomposition are closely related to the closability, singularity and continuity, respectively, of the following linear relation

$$\widehat{B} := \{ \{Ax, Bx\} \in \mathfrak{H}_A \times \mathfrak{H}_B : x \in \mathfrak{H} \},$$

acting between the model Hilbert spaces \mathfrak{H}_A and \mathfrak{H}_B . Note that in case when A and B fulfill the following kernel relation

$$(7.2) \quad \ker A \subseteq \ker B,$$

then \widehat{B} coincides with the quotient operator J_B^*/J_A^* in the sense of Izumino, see [26]. Nevertheless, \widehat{B}^{**} , i.e. the closure of \widehat{B} is not an operator in general: even under the extra condition (7.2) \widehat{B} may be maximally singular and in view of Theorem 7.8 this means just that A and B are singular, see Example 7.13 below.

Before proving the main theorem of this section we need the following lemma that is of certain interest in its own right:

Lemma 7.1. *Assume that A and B are bounded positive linear operators on a Hilbert space \mathfrak{H} . Assume also that T is a closable operator from \mathfrak{H}_A into \mathfrak{H}_B such that $\text{ran } A \subseteq \text{dom } T$. Then $(TJ_A^*)^*(TJ_A^*)$ is a bounded positive linear operator acting in \mathfrak{H} such that it is absolutely continuous with respect to A .*

Proof. First of all we notice that TJ_A^* is bounded according to the Banach closed graph theorem. Indeed, the conditions of the cited theorem are guaranteed by the identities

$$\operatorname{ran} J_A^* = \operatorname{ran} A \subseteq \operatorname{dom} T.$$

Put furthermore a sequence $\{x_n\}_{n=1}^\infty$ from \mathfrak{H} such that $(Ax_n, x_n) \rightarrow 0$ and that

$$((TJ_A^*)^*(TJ_A^*)(x_n - x_m), x_n - x_m) = \langle T(Ax_n - Ax_m), T(Ax_n - Ax_m) \rangle_B \rightarrow 0.$$

Then according to the closability of T it follows that

$$((TJ_A^*)^*(TJ_A^*)x_n, x_n) = \langle T(Ax_n), T(Ax_n) \rangle_B \rightarrow 0,$$

i.e. $(TJ_A^*)^*(TJ_A^*)$ is absolutely continuous with respect to A . \square

Theorem 7.2. *Let A and B be bounded positive linear operators on a complex Hilbert space \mathfrak{H} . Then*

$$(7.3) \quad \mathfrak{M} = \{\xi \in \mathfrak{H}_B : \exists \{x_n\}_{n=1}^\infty \text{ in } \mathfrak{H}, (Ax_n, x_n) \rightarrow 0, Bx_n \rightarrow \xi\}$$

is a closed linear subspace of \mathfrak{H}_B and if P stands for the orthogonal projection of \mathfrak{H}_B onto \mathfrak{M} then we have the following decomposition for B :

$$B = B_c + B_s,$$

*where $B_c = J_B^{**}(I - P)J_B^*$ is absolutely continuous with respect to A , and $B_s = J_B^{**}PJ_B^*$ is singular with respect to A . Moreover, if C is a positive linear operator such that C is absolutely continuous with respect to A and $C \leq B$ then $C \leq J_B^{**}(I - P)J_B^* = B_c$.*

Proof. In order to prove that $J_B^{**}(I - P)J_B^*$ is absolutely continuous with respect to A consider the following linear relation on $\mathfrak{H}_A \times \mathfrak{H}_B$:

$$(7.4) \quad \widehat{B} := \{\{Ax, Bx\} \in \mathfrak{H}_A \times \mathfrak{H}_B : x \in \mathfrak{H}\}.$$

According to [24, Theorem 4.1], the regular part of this relation, denoted by T , acts just as a closable operator between \mathfrak{H}_A and \mathfrak{H}_B with the domain $\operatorname{dom} T = \operatorname{ran} A$ as follows

$$Ax \mapsto (I - P)Bx, \quad x \in \mathfrak{H}.$$

From Lemma 7.1 we immediately obtain that $(TJ_A^*)^*(TJ_A^*)$ is absolutely continuous with respect to A . The following line of identities

$$\begin{aligned} ((TJ_A^*)^*(TJ_A^*)x, y) &= \langle TJ_A^*x, TJ_A^*y \rangle_B \\ &= \langle (I - P)Bx, (I - P)By \rangle_B \\ &= \langle (I - P)J_B^*x, J_B^*y \rangle_B \\ &= (J_B^{**}(I - P)J_B^*x, y) \end{aligned}$$

implies also that

$$(7.5) \quad J_B^{**}(I - P)J_B^* = (TJ_A^*)^*(TJ_A^*)$$

and thus $J_B^{**}(I - P)J_B^*$ is absolutely continuous with respect to A , as it is claimed.

In the next step, we prove that any A -absolute continuous positive linear operator C with $C \leq B$ also fulfills $C \leq J_B^{**}(I - P)J_B^*$. Notice that Douglas' factorization theorem [15, Theorem 1] and the inequality $C \leq J_B^{**}J_B^*$ yield a linear operator $\widehat{C} \in \mathcal{B}(\mathfrak{H}, \mathfrak{H}_B)$ fulfilling

$$(7.6) \quad C^{1/2} = J_B^{**}\widehat{C} = \widehat{C}^*J_B^*.$$

Furthermore, the fact that $\text{ran } B$ is dense in the model Hilbert space \mathfrak{H}_B and inequalities

$$(\widehat{C}^*(Bx), \widehat{C}^*(Bx)) = \|C^{1/2}x\|^2 \leq \|B^{1/2}x\|^2 = \langle Bx, Bx \rangle_B, \quad x \in \mathfrak{H}$$

imply that $\|\widehat{C}\| = \|\widehat{C}^*\| \leq 1$. On the other hand, in view of the following factorization

$$(7.7) \quad C = J_B^{**}\widehat{C}\widehat{C}^*J_B^*$$

we have to prove that the following inequality holds true:

$$\|\widehat{C}^*J_B^*x\|^2 \leq \langle (I - P)J_B^*x, (I - P)J_B^*x \rangle_B, \quad x \in \mathfrak{H}.$$

Equivalently we state the following inequality:

$$(7.8) \quad \|\widehat{C}^*\varphi\|^2 \leq \langle (I - P)\varphi, (I - P)\varphi \rangle_B, \quad \varphi \in \mathfrak{H}_B.$$

Since \widehat{C}^* is a contraction, this will be obtained by showing that $\text{ran } \widehat{C} \subseteq \text{ran}(I - P)$. Indeed, then also $\text{ran } P \subseteq \ker \widehat{C}^*$ is fulfilled so that (7.8) follows:

$$\|\widehat{C}^*\varphi\|^2 = \|\widehat{C}^*(I - P)\varphi\|^2 \leq \langle (I - P)\varphi, (I - P)\varphi \rangle_B,$$

holds for all $\varphi \in \mathfrak{H}_B$.

Taking into account the following identity

$$\overline{\text{ran}(\widehat{C}\widehat{C}^*J_B^*)} = \overline{\text{ran } \widehat{C}},$$

it suffices to prove that $\text{ran}(\widehat{C}\widehat{C}^*J_B^*) \subseteq \mathfrak{M}^\perp$. Let $\xi \in \mathfrak{M}$ and put a sequence $\{x_n\}_{n=1}^\infty$ from \mathfrak{H} such that $(Ax_n, x_n) \rightarrow 0$ and that $Bx_n \rightarrow \xi \in \mathfrak{H}_B$. Since $C \leq B$, we find that

$$(C(x_n - x_m), x_n - x_m) \leq (B(x_n - x_m), x_n - x_m) \rightarrow 0,$$

and consequently that $(Cx_n, x_n) \rightarrow 0$ via the A -absolute continuity of C . On the other hand,

$$\begin{aligned} \left| \langle \xi, \widehat{C}\widehat{C}^*J_B^*x \rangle_B \right|^2 &= \lim_{n \rightarrow \infty} \left| \langle Bx_n, \widehat{C}\widehat{C}^*J_B^*x \rangle_B \right|^2 \\ &= \lim_{n \rightarrow \infty} \left| \langle J_B^*x_n, \widehat{C}\widehat{C}^*J_B^*x \rangle_B \right|^2 \\ &= \lim_{n \rightarrow \infty} |(x_n, Cx)|^2 \\ &\leq \lim_{n \rightarrow \infty} (Cx_n, x_n)(Cx, x) = 0, \end{aligned}$$

for all $x \in \mathfrak{H}$.

It only remains to show that $J_B^{**}PJ_B^* = B - J_B^{**}(I - P)J_B^*$ is singular with respect to A . This follows from the maximality property of $J_B^{**}(I - P)J_B^*$ proved above: if C is a positive operator with $C \leq A$ and $C \leq J_B^{**}PJ_B^*$ then clearly $C + J_B^{**}(I - P)J_B^* \leq B$ so that it is absolutely continuous with respect to A and hence we have

$$C + J_B^{**}(I - P)J_B^* \leq J_B^{**}(I - P)J_B^*.$$

Thus $C = 0$ which completes the proof. \square

By using the notations of Ando [2] and Theorem 7.2 above we obtain the following result:

Corollary 7.3. *For any bounded positive linear operators A, B one has*

$$(7.9) \quad [A]B = J_B^{**}(I - P)J_B^*.$$

Proof. It follows from the fact that both $[A]B$ and $J_B^{**}(I - P)J_B^*$ are the largest among the set of positive operators C which are A -absolute continuous with $C \leq B$, see [2, Theorem 2]. \square

A natural representation for A -absolute continuous operators dominated by B appears in the next corollary:

Corollary 7.4. *For any A -absolute continuous positive operator $C \in \mathcal{B}(\mathfrak{H})$ with $C \leq B$ there is a unique positive operator $\tilde{C} \in \mathcal{B}(\mathfrak{H}_B)$ with $\|\tilde{C}\| \leq 1$ and $\text{ran } \tilde{C} \subseteq \mathfrak{M}^\perp$ such that*

$$C = J_B^{**}\tilde{C}J_B^*.$$

Proof. The existence of an operator with the prescribed properties has been shown in the proof of Theorem 7.2, namely $\tilde{C} = \widehat{C}\widehat{C}^*$. The uniqueness follows from the following identities

$$\langle \tilde{C}J_B^*x, J_B^*y \rangle_B = \langle Cx, y \rangle = \langle \widehat{C}\widehat{C}^*J_B^*x, J_B^*y \rangle_B, \quad x, y \in \mathfrak{H},$$

by noticing that $\text{ran } J_B^*$ is dense in \mathfrak{H}_B . \square

7.2. Characterizations of absolute continuity and singularity

First of all we characterize the domain of the adjoint \widehat{B}^* of the relation \widehat{B} that will be essential in the proof of the main theorem of this section. For more information about linear relations we refer to [4] and [24].

Lemma 7.5. *Let \widehat{B} denote the linear relation introduced in (7.4). Then*

$$\text{dom } \widehat{B}^* = \{\xi \in \mathfrak{H}_B : J_B^{**}\xi \in \text{ran } A^{1/2}\}.$$

Proof. According to [24, Lemma 9.1], $\text{dom } \widehat{B}^*$ consists of vectors $\xi \in \mathfrak{H}_B$ for them

$$|\langle Bx, \xi \rangle_B|^2 \leq m_\xi \langle Ax, Ax \rangle_A,$$

for all $x \in \mathfrak{H}$ holds with some nonnegative constant m_ξ . According to identities $Ax = J_A^*x$ and $Bx = J_B^*x$ this can be reformulated as follows

$$|(x, J_B^{**}\xi)|^2 \leq m_\xi \langle J_A^*x, J_A^*x \rangle_A,$$

for all $x \in \mathfrak{H}$. That means that $J_B^{**}\xi$ belongs to $\text{ran } J_A^{**} = \text{ran } A^{1/2}$ according to [44, Theorem 1]. \square

The following theorem gives a characterization of A -absolute continuous operators. The equivalence of (i) and (v) below is due to Ando [2, Theorem 5].

Theorem 7.6. *Let $A, B \in \mathcal{B}(\mathfrak{H})$ be positive linear operators. The following statements are equivalent:*

- (i) B is absolutely continuous with respect to A .
- (ii) $\mathfrak{M} = \{0\}$.
- (iii) The linear relation \widehat{B} in (7.4) defines a closable operator from \mathfrak{H}_A into \mathfrak{H}_B .
- (iv) The set $\{\xi \in \mathfrak{H}_B : J_B^{**}\xi \in \text{ran } A^{1/2}\}$ is dense in \mathfrak{H}_B .
- (v) The set $\{x \in \mathfrak{H} : B^{1/2}x \in \text{ran } A^{1/2}\}$ is dense in \mathfrak{H} .

If any of the above conditions is satisfied and if \widehat{B} stands for the mapping defined in (7.4) then we have the following factorization for B :

$$(7.10) \quad B = (\widehat{B}J_A^*)^*(\widehat{B}J_A^*).$$

Proof. It is easy to check from the definition that the A -absolute continuity of B is equivalent to the closability of \widehat{B} . Since \mathfrak{M} coincides with the multivalued part of the closure of \widehat{B} , the equivalence of (ii) and (iii) is also clear. Finally, according to [24, Lemma 9.1] a linear relation is (the graph of a) closable operator if and only if its adjoint is densely defined. Thus the equivalence of (iii) and (iv) follows due to Lemma 7.5.

It only remains to show that (iv) and (v) are equivalent. In order to show that let V denote the following isometry from $\text{ran } B$ of \mathfrak{H}_B into \mathfrak{H} :

$$Bx \mapsto B^{1/2}x, \quad x \in \mathfrak{H},$$

so that the unique continuous extension V^{**} of V acts also isometric between \mathfrak{H}_B and \mathfrak{H} with the range $\overline{\text{ran } A}$. According to the characteristic extension property of J_B^* we obtain the following identity:

$$(7.11) \quad VJ_B^* = J_B^{**}V^* = B^{1/2}.$$

In account of the surjectivity of V^* , by using (7.11) we have the following identities:

$$\begin{aligned}\{\xi \in \mathfrak{H}_B : J_B^{**}\xi \in \text{ran } A^{1/2}\} &= \{V^*x : x \in \mathfrak{H}, J_B^{**}V^*x \in \text{ran } A^{1/2}\} \\ &= \{V^*x : x \in \mathfrak{H}, B^{1/2}x \in \text{ran } A^{1/2}\}.\end{aligned}$$

Thus, by using the fact that $V^{**}V^*$ coincides with the orthogonal projection Q of \mathfrak{H} onto $\overline{\text{ran } A}$, we obtain that statement (iv) holds if and only if the set

$$\{Qx : x \in \mathfrak{H}, B^{1/2}x \in \text{ran } A^{1/2}\}$$

is dense in $\overline{\text{ran } A}$ and this is obviously equivalent to statement (v). \square

In the next theorem we give another characterization of absolute continuity. We remark that the original definition of absolute continuity given by Ando [2] is just property (ii) below. Furthermore, an operator B that satisfies (ii) is also used to be called A -almost dominated, cf. [25].

Theorem 7.7. *Assume that A and B are bounded positive linear operators on a Hilbert space \mathfrak{H} . The following statements are equivalent:*

- (i) *B is absolutely continuous with respect to A .*
- (ii) *There are nondecreasing sequences of positive operators $\{B_n\}_{n=1}^\infty$ and of positive numbers $\{\alpha_n\}_{n=1}^\infty$ such that $B_n \leq \alpha_n A$ and that $B_n x \rightarrow Bx$ for all $x \in \mathfrak{H}$.*

Proof. Assume first that B is A -absolute continuous. Due to Theorem 7.6 we find that $B = (\widehat{B}J_A^*)^* \widehat{B}J_A^*$ where \widehat{B} is the closable operator introduced by formula (7.4). Let S stand for the square root of the (unbounded) positive selfadjoint operator $\widehat{B}^* \widehat{B}^{**}$. Then S acts in the model Hilbert space \mathfrak{H}_A with the domain $\text{dom } S = \text{dom } \widehat{B}^{**}$ so that $B = (SJ_A^*)^* SJ_A^*$. Let E stand for the spectral measure of S and for any $n \in \mathbb{N}$ define

$$S_n = \int_0^n id_{\mathbb{R}} dE,$$

so that $\{S_n\}_{n=1}^\infty$ is a nondecreasing sequence of bounded positive operators acting on \mathfrak{H}_A . For any $n \in \mathbb{N}$ define

$$B_n = J_A^{**}(S_n)^2 J_A^* = (S_n J_A^*)^* S_n J_A^*.$$

It is clear, that $\{B_n\}_{n=1}^\infty$ is a nondecreasing sequence of positive operators on \mathfrak{H} . On the other hand, for any $n \in \mathbb{N}$ we conclude that

$$(B_n x, x) = \langle S_n J_A^* x, S_n J_A^* x \rangle_A \leq \|S_n\|^2 (Ax, x),$$

for all $x \in \mathfrak{H}$, i.e. $B_n \leq \|S_n\|^2 A$. It only remains to show that $\{B_n\}_{n=1}^\infty$ converges strongly to B . Since $\{B_n\}_{n=1}^\infty$ is a nondecreasing sequence of positive operators such that $B_n \leq B$ for all $n \in \mathbb{N}$, we conclude that it converges strongly to a bounded positive operator.

Hence it suffices to show that $\{B_n\}_{n=1}^\infty$ converges to B in the weak operator topology: by putting $x, y \in \mathfrak{H}$ we obtain that $S_n J_A^* x \rightarrow S J_A^* x$ and $S_n J_A^* y \rightarrow S J_A^* y$. Consequently,

$$\begin{aligned} (Bx, y) &= \langle S J_A^* x, S J_A^* y \rangle_A = \lim_{n \rightarrow \infty} \langle S_n J_A^* x, S_n J_A^* y \rangle_A = \lim_{n \rightarrow \infty} (J_A^{**}(S_n)^2 J_A^* x, y) \\ &= \lim_{n \rightarrow \infty} (B_n x, y), \end{aligned}$$

as it is claimed.

Conversely, if $\{B_n\}_{n=1}^\infty$ is a sequence with the desired properties then B_n is A -absolute continuous and $B_n \leq B$ for any $n \in \mathbb{N}$. Therefore, $B_n \leq J_B^{**}(I - P)J_B^*$ according to Theorem 7.2 and thus

$$(Bx, x) = \lim_{n \rightarrow \infty} (B_n x, x) \leq (J_B^{**}(I - P)J_B^* x, x), \quad x \in \mathfrak{H}.$$

Consequently, $B = J_B^{**}(I - P)J_B^*$, i.e. B is A -absolute continuous. \square

In the next theorem we characterize the A -singular positive operators:

Theorem 7.8. *Let $A, B \in \mathcal{B}(\mathfrak{H})$ be positive operators. The following statements are equivalent:*

- (i) B is singular with respect to A .
- (ii) $\mathfrak{M} = \mathfrak{H}_B$.
- (iii) The linear relation \widehat{B} is maximally singular, i.e. $\text{dom } \widehat{B}^* = \{0\}$.
- (iv) $J_B^{**}\xi \in \text{ran } A^{1/2}$ for some $\xi \in \mathfrak{H}_B$ implies $\xi = 0$.
- (v) $\text{ran } A^{1/2} \cap \text{ran } B^{1/2} = \{0\}$.

Proof. Assume first that B is A -singular. According to Theorem 7.2, $J_B^{**}(I - P)J_B^*$ is A -absolute continuous and thus, in view of Theorem 7.7, there exists a nondecreasing sequence $\{C_n\}_{n=1}^\infty$ of positive operators that converges strongly to $J_B^{**}(I - P)J_B^*$ so that $C_n \leq \alpha_n A$ for some $\alpha_n \geq 1$ holds. Then for any integer n we have $\alpha_n^{-1} C_n \leq B$ and $\alpha_n^{-1} C_n \leq A$, and therefore $C_n = 0$ in account of the A -singularity of B . Hence we have $J_B^{**}(I - P)J_B^* = 0$, which implies

$$0 = \langle (I - P)J_B^* x, (I - P)J_B^* x \rangle_B = \langle (I - P)(Bx), (I - P)(Bx) \rangle_B,$$

for all $x \in \mathfrak{H}$. Consequently, $P = I$, i.e. $\mathfrak{M} = \mathfrak{H}_B$ and therefore, (i) implies (ii). The reverse implication is obvious. Note that

$$\mathfrak{M} = \text{mul } \widehat{B}^{**} = \{\text{dom } \widehat{B}^*\}^\perp,$$

according to the definition of \mathfrak{M} , where $\text{mul } \widehat{B}^{**}$ stands for the multivalued part of the relation \widehat{B}^{**} , see [24]. Hence the equivalence of (ii) and (iii) is also clear. Finally, according to Lemma 7.5 we have

$$\text{dom } \widehat{B}^* = (J_B^{**})^{-1}(\text{ran } A^{1/2}).$$

Since J_B^{**} is injective, the equivalence of (iii), (iv) and (v) is also clear. \square

7.3. Uniqueness of the decomposition

Contrary to the case of measures, the Lebesgue-type decomposition among positive operators is not necessarily unique. However, a necessary and sufficient condition guaranteeing uniqueness in the decomposition was given in [2, Theorem 6] by T. Ando. The purpose of the present section is to revise Ando's result: it turns out that the uniqueness is up to the continuity of the regular part of the linear relation \widehat{B} . We also refer to [25, Theorem 4.6] of Hassi et al. for an analog version of Ando's uniqueness theorem for forms.

Theorem 7.9. *Let A and B be positive linear operators in the Hilbert space \mathfrak{H} . The following assertions are equivalent:*

- (i) *The Lebesgue-type decomposition of B into A -absolute continuous and A -singular parts is unique.*
- (ii) *$\text{dom } \widehat{B}^* \subseteq \mathfrak{H}_B$ is closed.*
- (iii) *$J_B^{**}(\mathfrak{M}^\perp) \subseteq \text{ran } A^{1/2}$.*
- (iv) *The regular part of \widehat{B} , i.e. the linear operator $\mathfrak{H}_A \supseteq \text{ran } A \rightarrow \mathfrak{H}_B$ defined by the following correspondence*

$$(7.12) \quad Ax \mapsto (I - P)Bx, \quad x \in \mathfrak{H},$$

is continuous.

- (v) *$B_c := J_B^{**}(I - P)J_B^*$ is dominated by A , i.e. $B_c \leq \alpha A$ for some nonnegative constant α .*

Proof. Assume first that $\text{dom } \widehat{B}^*$ is not closed. Since $\{\text{dom } \widehat{B}^*\}^\perp = \mathfrak{M}$, this means that there exists a (non-zero) vector $\zeta \in \mathfrak{M}^\perp \setminus \text{dom } \widehat{B}^*$. Let Q denote the orthogonal projection of \mathfrak{H}_B onto the one-dimensional subspace spanned by ζ . Since $\text{ran } Q \subseteq \mathfrak{M}^\perp$, this implies that P and Q are orthogonal to each other, and thus, $P + Q$ is also an orthogonal projection. Now let us define the following positive linear operators acting in the original Hilbert space \mathfrak{H} as follows:

$$(7.13) \quad B_1 := J_B^{**}(I - P - Q)J_B^*, \quad B_2 := J_B^{**}(P + Q)J_B^*.$$

Then clearly, $B_1 + B_2 = B$. We state that B_1 is A -absolute continuous and B_2 is A -singular.

In the proof of Theorem 7.2 we saw that the mapping $T : \mathfrak{H}_A \supseteq \text{ran } A \rightarrow \mathfrak{H}_B$ defined by the correspondence (7.12) above is closable. Since Q is an orthogonal projection with one-dimensional range, we obtain that $(I - Q)T$ is closable too (see [25, Proposition 4.3]). Thus, according to Lemma 7.1 it follows that $((I - Q)TJ_A^*)^*(I - Q)TJ_A^*$ acts on \mathfrak{H} as a bounded positive linear operator that is absolutely continuous with respect to A . On the

other hand, for any $x, y \in \mathfrak{H}$ we find that

$$\begin{aligned} ((I - Q)TJ_A^*)^*(I - Q)TJ_A^*x, y) &= \langle (I - Q)T(Ax), (I - Q)T(Ay) \rangle_B \\ &= \langle (I - Q)(I - P)(Bx), (I - Q)(I - P)(By) \rangle_B \\ &= \langle (I - P - Q)J_B^*x, J_B^*y \rangle_B \\ &= (B_1x, y), \end{aligned}$$

i.e. $B_1 = ((I - Q)TJ_A^*)^*(I - Q)TJ_A^*$. Therefore, B_1 is A -absolute continuous.

Our next claim is to show the A -singularity of B_2 . In view of Theorem 7.8 it is enough (and also necessary) to prove that

$$(7.14) \quad \text{ran } B_2^{1/2} \cap \text{ran } A^{1/2} = \text{ran } J_B^{**}(Q + P) \cap \text{ran } A^{1/2} = \{0\}.$$

So assume that $J_B^{**}(P + Q)\xi \in \text{ran } A^{1/2}$ for some $\xi \in \mathfrak{H}_B$. According to Lemma 7.5 this means that $(P + Q)\xi \in \text{dom } \widehat{B}^*$. On the other hand, we have $\text{dom } \widehat{B}^* \subseteq \mathfrak{M}^\perp$ and therefore

$$0 = \langle (P + Q)\xi, P\xi \rangle_B = \langle P\xi, P\xi \rangle_B.$$

Consequently,

$$(P + Q)\xi = Q\xi \in \text{dom } \widehat{B}^* \cap \text{ran } Q = \{0\},$$

and this implies (7.14) as well.

It only remains to prove that the pair (B_1, B_2) differs from (B_c, B_s) , i.e. $J_B^{**}QJ_B^* \neq 0$. (Here, $B_c = J_B^{**}(I - P)J_B^*$ and $B_s = B - B_c$.) Nevertheless, we have $\|J_B^{**}QJ_B^*\| = \|J_B^{**}Q\|^2$ on the one hand, and

$$J_B^{**}Q\zeta = J_B^{**}\zeta \neq 0$$

on the other hand due to the injectivity of J_B^{**} . Thus, $J_B^{**}QJ_B^* \neq 0$, indeed.

The implications (ii) \Rightarrow (iii) and (iii) \Rightarrow (iv) are obtained easily from the Banach closed graph theorem by only noticing that $\text{dom } T^* = \text{dom } \widehat{B}^* \oplus \mathfrak{M}$ (see [24, Proposition 9.2]) and that $\text{dom } \widehat{B}^* = \mathfrak{M}^\perp$. If the linear operator T defined in (7.12) is continuous, then we have

$$(J_B^{**}(I - P)J_B^*x, x) = \langle T(Ax), T(Ax) \rangle_B \leq \|T\|^2 \langle Ax, Ax \rangle_A = \|T\|^2 (Ax, x)$$

for any $x \in \mathfrak{H}$ and thus (iv) implies (v). Finally, assume B_c to be dominated by A in the sense of (v) and (B_1, B_2) to be a pair of positive linear operators satisfying $B_1 + B_2 = B$ such that B_1 is A -absolute continuous and B_2 is A -singular. Then from Theorem 7.2 we obtain that $B_1 \leq B_c$, and hence

$$0 \leq B_c - B_1 \leq B_c \leq \alpha A.$$

This implies that

$$0 \leq B_2 - B_s = (B - B_1) - (B - B_c) \leq B_c \leq \alpha A \quad \text{and} \quad B_2 - B_s \leq B_2,$$

therefore $B_2 - B_s = 0$, according to the A -singularity of B_2 . Consequently, $B_1 = B_c$ as well and the Lebesgue-type decomposition is unique. \square

7.4. Further remarks and examples

From Lemma 7.1 and Theorem 7.6 we obtain that a positive operator $B \in \mathcal{B}(\mathfrak{H})$ is A -absolute continuous if and only if there exists a closable operator T from \mathfrak{H}_A into \mathfrak{H}_B with $\text{ran } A \subseteq \text{dom } T$ such that

$$(7.15) \quad B = (TJ_A^*)^*(TJ_A^*).$$

On the right side of formula (7.15) we may not write $J_A^{**}T^*TJ_A^*$ in general, just whenever $\text{ran } B \subseteq \text{ran } A^{1/2}$ holds. In view of equality $\text{ran } A^{1/2} = \text{ran } J_A^{**}$, the necessity of this range inclusion is obvious. The conversion of this assertion is stated in the next theorem:

Theorem 7.10. *Assume that A and B are positive linear operators in the Hilbert space \mathfrak{H} such that $\text{ran } B \subseteq \text{ran } A^{1/2}$. Then B is absolutely continuous with respect to A and the following factorization for B holds:*

$$(7.16) \quad B = J_A^{**}\widehat{B}^*\widehat{B}J_A^*$$

Proof. In order to prove that B is A -absolute continuous we show that the dense linear manifold $\text{ran } B$ is contained by $\text{dom } \widehat{B}^*$ so that condition (iv) of Theorem 7.6 satisfies. Indeed, for any $x \in \mathfrak{H}$ we have that

$$J_B^{**}(Bx) = Bx \in \text{ran } A^{1/2},$$

thus $\mathfrak{H}_B \ni Bx$ belongs to $\text{dom } \widehat{B}^*$. On the other hand, for any $x \in \mathfrak{H}$ we find that

$$\widehat{B}J_A^*x = \widehat{B}(Ax) = Bx \in \text{dom } \widehat{B}^*.$$

Since $(\widehat{B}J_A^*)^*$ extends $J_A^{**}\widehat{B}^*$, this yields just identity (7.16). \square

We say that the positive operator B is *dominated* by A if there is a nonnegative constant α such that $B \leq \alpha A$. Due to the Douglas factorization theorem [15, Theorem 1] this is equivalent to the range inclusion $\text{ran } B^{1/2} \subseteq \text{ran } A^{1/2}$. Taking in to account the previous theorem, it is easy to check that a positive operator dominated by A is automatically A -absolute continuous. However, as the following example shows, the conversion of this statement is not true in general:

Example 7.11. Let A be a bounded positive linear operator on the Hilbert space \mathfrak{H} . According to the Schwarz-inequality

$$(A^2x, x) \leq \|A\|(Ax, x), \quad x \in \mathfrak{H},$$

it is clear that $B := A^2$ is dominated by A (and hence also A -absolute continuous). Conversely, A is always absolutely continuous with respect to A^2 due to Theorem 7.10,

nevertheless, A is dominated by A^2 just whenever A has closed range, see [61, Theorem 3.1]. Furthermore, the mapping $\widehat{B} : \mathfrak{H}_{A^2} \rightarrow \mathfrak{H}_A$ defined by the relation

$$A^2x \mapsto Ax, \quad x \in \mathfrak{H},$$

defines a closable operator satisfying $A = J_{A^2}^{**} \widehat{B}^* \widehat{B} J_{A^2}^*$.

As the following example shows, $\text{ran } B \subseteq \text{ran } A^{1/2}$ is not a necessary condition of absolute continuity:

Example 7.12. Let A be positive linear operator on a Hilbert space \mathfrak{H} with dense range. Then the identity operator I of \mathfrak{H} is A -absolute continuous but the range inclusion $\text{ran } I \subseteq \text{ran } A^{1/2}$ holds exactly when A has bounded inverse. Indeed, our assumptions imply that A is injective so that its inverse A^{-1} acts as a positive selfadjoint operator in \mathfrak{H} with $\text{dom } A^{-1} = \text{ran } A$. The A -absolute continuity of I is easily obtained from the fact that $A^{-1/2}$, the inverse of $A^{1/2}$ is closed.

It is easy to check that the relation \widehat{B} is an operator (i.e. the graph of an operator) if and only if the kernel condition (7.2) is fulfilled. However, the following example shows that \widehat{B} may be non-closable also under this extra condition, that is (7.2) does not guarantee the A -absolute continuity of B . Moreover, B may be A -singular also in that case:

Example 7.13. Let A_0 be any positive linear operator on \mathfrak{H} with non-closed range. According to [61, Theorem 2.5], there is a $y \in \text{ran } A_0^{1/2} \setminus \text{ran } A_0$. Let $A = A_0^2$ and let B be the orthogonal projection of \mathfrak{H} onto the one-dimensional subspace spanned by y . Then one easily verifies that the kernel condition $\ker A \subseteq \ker B$ is satisfied, i.e. \widehat{B} is an operator. Nevertheless, we state that B is A -singular, i.e. $\text{dom } \widehat{B}^* = \{0\}$. To see this, fix a $\xi \in \text{dom } \widehat{B}^*$. Due to Lemma 7.5 this means that $J_B^{**}\xi \in \text{ran } A^{1/2} = \text{ran } A_0$. But we have $\text{ran } J_B^{**} = \text{ran } B$ and that $\text{ran } B \cap \text{ran } A_0 = \{0\}$, thus $J_B^{**}\xi = 0$. Since $\ker J_B^{**} = \{0\}$, we have $\xi = 0$, as it is claimed.

We notice that $\widehat{B} : \mathfrak{H}_A \rightarrow \mathfrak{H}_B$ from the previous example is also an example for a densely defined operator which is maximally singular, cf. also [31] and [38].

We close this chapter with a characterization of positive linear operators with closed range in terms of the uniqueness of the Lebesgue-type decomposition, cf. [2, Corollary 7]:

Theorem 7.14. *Let A be a positive linear operator on the Hilbert space \mathfrak{H} . The following statements are equivalent:*

- (i) A has closed range.
- (ii) Any positive linear operator B admits a unique Lebesgue-type decomposition with respect to A .

Proof. Assume first that $\text{ran } A$ is closed and fix a positive linear operator B . From [61, Theorem 2.6] we obtain that

$$\text{ran } J_A^* = \text{ran } A \subseteq \mathfrak{H}_A$$

is closed. Thus the operator defined by (7.12) is defined on the whole of \mathfrak{H}_A and therefore it is bounded according to the Banach closed graph theorem. Consequently, the Lebesgue-type decomposition of B into A -absolute continuous and A -singular parts is unique due to Theorem 7.9. Conversely, if the range of A is not closed then $B := A^{1/2}$ is A -absolute continuous but not A -dominated according to Theorem 7.10 and Example 7.11, respectively. Thus, according again to Theorem 7.9, the Lebesgue-type decomposition of B is not unique. \square

Bibliography

- [1] W. N. Anderson, Jr. and G. E. Trapp, Shorted Operators. II., *SIAM J. Appl. Math.*, **28** (1975), 60–71.
- [2] T. Ando, Lebesgue-type decomposition of positive operators, *Acta Sci. Math. (Szeged)*, **38** (1976), 253–260.
- [3] T. Ando and K. Nishio, Positive selfadjoint extensions of positive symmetric operators, *Tôhoku Math. J.*, **22** (1970), 65–75.
- [4] R. Arens, Operational calculus of linear relations, *Pacific J. Math.*, **11** (1961), 9–23.
- [5] Y. M. Arlinskiĭ, S. Hassi, Z. Sebestyén, and H. S. V. de Snoo, On the class of extremal extensions of a nonnegative operator, In *Recent advances in operator theory and related topics (Szeged, 1999)*, Oper. Theory Adv. Appl. **127**, Birkhauser (Basel, 2001), 41–81.
- [6] Y. M. Arlinskiĭ and E. R. Tsekanovskii, Quasi selfadjoint contractive extensions of Hermitian contractions, *Teor. Funkts., Funkts. Anal. Prilozhen*, **50** (1988), 9–16.
- [7] S. Banach, *Théorie des opérations linéaires*, Éditions Jacques Gabay, Sceaux, 1993. Reprint of the 1932 original.
- [8] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses. Theory and Applications*. Springer, New York, 2003. Second edition.
- [9] F. J. Beutler, *The operator theory of the pseudoinverse. I. Bounded operators*, J. Math. Anal. Appl., **10** (1965), 451–470.
- [10] F. J. Beutler, *The operator theory of the pseudoinverse. II. Unbounded operators with arbitrary range*, J. Math. Anal. Appl., **10** (1965), 471–493.
- [11] F. E. Browder, Functional analysis and partial differential equations. I, *Math. Ann.*, **138** (1959), 55–79.
- [12] J. Dieudonné, Quasi-hermitian operators, In *Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960)*, pages 115–122. Jerusalem Academic Press, Jerusalem, 1961.
- [13] J. Dieudonné, *Treatise on analysis. Vol. II*, Translated from the French by I. G. Macdonald. Pure and Applied Mathematics, Vol. 10-II. Academic Press, New York, 1970.
- [14] J. Dixmier, Étude sur les variétés et les opérateurs de Julia, avec quelques applications, *Bull. Soc. Math. Fr.*, **77** (1949), 11–101.
- [15] R. G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert space, *Proc. Amer. Math. Soc.*, **17** (1966), 413–415.
- [16] B. Farkas and M. Matolcsi, Commutation properties of the form sum of positive, symmetric operators, *Acta Sci. Math (Szeged)*, **67** (2001), 777–790.
- [17] P. A. Fillmore and J. P. Williams, On operator ranges, *Advances in Math.*, **7** (1971), 254–281.
- [18] K. Friedrichs, Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren, *Math. Ann.*, **109** (1934), 465–487.
- [19] И. Ц. Гохберг и М. Г. Крейн. Основные положения о дефектных числах, корневых числах и индексах линейных операторов. *УМН.*, **12** (1957), 43–118.

- [20] K. Gustafson, The RKNG (Rellich, Kato, Sz.-Nagy, Gustafson) perturbation theorem for linear operators in Hilbert and Banach space, *Acta Sci. Math. (Szeged)*, **45** (1983), 201–211.
- [21] S. Hassi, A. Sandovici, H.S.V. de Snoo, and H. Winkler, Form sums of nonnegative selfadjoint operators, *Acta Math. Hungar.*, **111** (2006), 81–105.
- [22] S. Hassi, A. Sandovici, H.S.V. de Snoo, and H. Winkler, Extremal extensions for the sum of non-negative selfadjoint relations, *Proc. Amer. Math. Soc.*, **135** (2007), 3193–3205.
- [23] S. Hassi, A. Sandovici, H.S.V. de Snoo, and H. Winkler, A general factorization approach to the extension theory of nonnegative operators and relations, *J. Operator Theory*, **58** (2007) 351–386.
- [24] S. Hassi, Z. Sebestyén, H. S. V. de Snoo, and F. H. Szafraniec, A canonical decomposition for linear operators and linear relations, *Acta Math. Hungar.*, **115** (2007), 281–307.
- [25] S. Hassi, Z. Sebestyén, and H. de Snoo, Lebesgue type decompositions for nonnegative forms, *J. Funct. Anal.*, **257** (2009), 3858–3894.
- [26] S. Izumino, Quotients of bounded operators, *Proc. Amer. Math. Soc.* **106** (1989), 427–435.
- [27] J. Joichi, On closed operators with closed range, *Proc. Am. Math. Soc.*, **11** (1960), 80–83.
- [28] L. Kapos and Z. Sebestyén, On range characterization of adjoint operators on Hilbert space, *Studia Sci. Math. Hungar.*, **30** (1995), 261–263.
- [29] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, second edition, 1976. Grundlehren der Mathematischen Wissenschaften, Band 132.
- [30] T. Kato, Perturbation theory for nullity, deficiency and other quantities of linear operators, *J. Analyse Math.*, **6** (1958), 261–322.
- [31] V. D. Koshmanenko and S. Ôta, On the characteristic properties of singular operators, *Ukrain. Mat. Zh.*, **48** (1996), 1484–1493.
- [32] М. Г. Крейн, Теория самосопряженных расширений полуограниченных эрмитовых операторов и её приложения. I, *Матем. Сб.*, **20** (1947), 431–495.
- [33] М. Г. Крейн, Теория самосопряженных расширений полуограниченных эрмитовых операторов и её приложения. II, *Матем. Сб.*, **21** (1947), 365–404.
- [34] M. G. Krein, Compact linear operators on functional spaces with two norms, *Integral Equations Operator Theory*, **30** (1998), 140–162. Translated from the Ukrainian, Dedicated to the memory of Mark Grigorievich Krein (1907–1989).
- [35] P. D. Lax, Symmetrizable linear transformations, *Comm. Pure Appl. Math.*, **7** (1954), 633–647.
- [36] J. v. Neumann, Allgemeine Eigenwerttheorie hermitescher Funktionaloperatoren, *Mathematische Annalen*, **102** (1930), 49–131.
- [37] J. v. Neumann, Über adjungierte Funktionaloperatoren, *The Annals of Mathematics*, **33** (1932), 294–310.
- [38] S. Ôta, On a singular part of an unbounded operator, *Z. Anal. Anwendungen*, **7** (1988), 15–18.
- [39] V. Prokaj and Z. Sebestyén, On Friedrichs extensions of operators, *Acta Sci. Math. (Szeged)*, **62** (1996), 243–246.
- [40] V. Prokaj and Z. Sebestyén, On extremal positive operator extensions, *Acta Sci. Math. (Szeged)*, **62** (1996), 485–491.
- [41] W. T. Reid, Symmetrizable completely continuous linear transformations in Hilbert space, *Duke Math. J.*, **18** (1951), 41–56.
- [42] F. Rellich, Störungstheorie der Spektralzerlegung. III, *Math. Ann.* **116** (1939), 555–570.
- [43] G. C. Rota, Extension theory of differential operators. I, *Comm. Pure Appl. Math.*, **11** (1958), 23–65.
- [44] Z. Sebestyén, On ranges of adjoint operators in Hilbert space, *Acta Sci. Math.*, **46** (1983), 295–298.
- [45] Z. Sebestyén, Restrictions of positive operators, *Acta Sci. Math. (Szeged)*, **46** (1983), 299–301.

- [46] Z. Sebestyén, Operator extensions on Hilbert space, *Acta Sci. Math. (Szeged)*, **57** (1993), 233–248.
- [47] Z. Sebestyén, Positivity of operator products, *Acta Sci. Math. (Szeged)*, **66** (2000), 287–294.
- [48] Z. Sebestyén and E. Sikolya, On Krein-von Neumann and Friedrichs extensions, *Acta Sci. Math. (Szeged)*, **69** (2003), 323–336.
- [49] Z. Sebestyén and J. Stochel, Restrictions of positive selfadjoint operators, *Acta Sci. Math. (Szeged)*, **55** (1991), 149–154.
- [50] Z. Sebestyén and J. Stochel, On products of unbounded operators, *Acta Math. Hungar.*, **100** (2003), 105–129.
- [51] Z. Sebestyén and J. Stochel, On suboperators with codimension one domains, *Journal of Mathematical Analysis and Applications*, **360** (2009), 391–397.
- [52] Z. Sebestyén and Zs. Tarsay, Biorthogonal expansions for symmetrizable operators, *Acta Math. Hungar.*, **131** (2011), 35–45.
- [53] Z. Sebestyén and Zs. Tarsay, T^*T always has a positive selfadjoint extension, *Acta Math. Hungar.*, **135** (2012), 116–129.
- [54] Z. Sebestyén and Zs. Tarsay, Characterizations of selfadjoint operators, *Manuscript*, 2013.
- [55] Z. Sebestyén and Zs. Tarsay, Characterizations of essentially skew/self-adjoint operators, *Manuscript*, 2013.
- [56] Z. Sebestyén and T. Titkos, Complement of forms, *To appear in Positivity*.
- [57] B. Simon, A canonical decomposition for quadratic forms with applications to monotone convergence theorems, *J. Funct. Anal.*, **28** (1978) 377–385.
- [58] M. H. Stone, *Linear transformations in Hilbert space*, American Mathematical Society Colloquium Publications, 15. (American Mathematical Society, Providence, Rhode Island: 1932.)
- [59] M. H. Stone, On One-Parameter Unitary Groups in Hilbert Space, *Ann. Math.*, **33** (1932), 643–648.
- [60] Zs. Tarsay, On operators with closed ranges, *Acta Sci. Math. (Szeged)*, **77** (2011), 579–587.
- [61] Zs. Tarsay, Operator extensions with closed range, *Acta Math. Hungar.*, **135** (2012), 325–341.
- [62] Zs. Tarsay, Lebesgue-type decomposition of positive operators, *Positivity*, online first (2012); DOI: 10.1007/s11117-012-0206-4
- [63] Zs. Tarsay, On form sums of positive operators, *Acta Math. Hungar.*, online first (2013); DOI: 10.1007/s10474-013-0299-7
- [64] J. Weidmann, *Linear operators in Hilbert spaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1980.
- [65] A. C. Zaenen, Über vollstetige symmetrische und symmetrisierbare Operatoren, *Nieuw Arch. Wiskunde*, **22** (1943), 57–80.
- [66] A. C. Zaenen, Normalisable transformations in Hilbert space and systems of linear integral equations, *Acta Math.*, **83** (1950), 197–248.
- [67] A. C. Zaenen, *Linear analysis. Measure and integral, Banach and Hilbert space, linear integral equations*, Interscience Publishers Inc., New York, 1953.

Summary

Symmetric, selfadjoint and semi-bounded operators play a highly important role in the theory of Hilbert space operators. The purpose of this thesis is to demonstrate some characterization-, factorization-, and extension-type results of the author which are related to these classes of operators. The dissertation is taken from the author's papers [52, 53, 60, 61, 62, 63]. The results of Chapter 1 and 2 are taken from the manuscripts [54, 55] of the author.

The notions of symmetricity and selfadjointness coincide among everywhere defined continuous operators. For unbounded operators the concept of selfadjointness is much stronger than symmetry. The difference between these concepts is illustrated by an example of Naimark for a closed symmetric operator whose square is a trivial operator, i.e. it is defined only on the null vector. Contrarily, according to a classical result of Neumann, the square of a selfadjoint operator is selfadjoint too, therefore obviously densely defined. The most effective tool of the theory of unbounded operators is the spectral theorem of Neumann. It also demonstrates that nontrivial operator calculus only among normal, in particular selfadjoint operators can be done.

The results of the theory of selfadjoint operators has a rich literature. Here we refer only to J. von Neumann [36] who characterized selfadjointness and the selfadjoint extendibility of densely defined closed symmetric operators by investigating their deficiency indices. In Chapters 1 and 2 we generalize Neumann's results. The strength of our results on the one hand is in that we do not need to assume that the operator in question is densely defined or closed, these properties are consequences of our assumptions. On the other hand, while the Cayley transform, the tool used by Neumann could only be employed on complex Hilbert spaces, for densely defined symmetric operators. Our method is applicable for the real and complex cases simultaneously. As an application of our results we prove the Kato-Rellich and Wüst perturbation theorems on real or complex Hilbert spaces.

A celebrated theorem of J. von Neumann reads as follows: if T is any densely defined closed operator between Hilbert spaces then T^*T is a positive selfadjoint operator. In Chapter 3 we prove that T^*T always has a positive selfadjoint extension. We also give some necessary and sufficient conditions on selfadjointness of T^*T , by giving a sharpening of the classical result of Friedrichs [18].

Chapter 4 considers operator extensions with closed range. We characterize those suboperators which admit a positive selfadjoint extension with closed range. We construct the Moore–Penrose pseudoinverse of the Krein–von Neumann extension.

In Chapter 5, continuing the work of Farkas and Matolcsi [16] we give a simple construction for the form sum of positive operators. Our approach makes it possible to give the domain, kernel and range characterizations for the form sum and its square root. In addition, we establish a criterion for the closedness of the range of the form sum. We give the Moore–Penrose pseudoinverse in this case.

In Chapter 6 an extension of a classical result due to Krein [34] on biorthogonal expansions of compact operators which are symmetrizable with respect to a nondegenerate positive operator, is given.

Finally, in Chapter 7 Lebesgue-type decomposition of bounded positive operators are investigated from a new view point, i.e. via operator factorizations and using the theory of linear relations.

Magyar nyelvű összefoglalás

A Hilbert terek lineáris operátorai közt kiemelkedően fontos szerepe van a szimmetrikus, az önadjungált, illetve a félig korlátos operátoroknak. Jelen dolgozat célja néhány a szerzőtől származó, ezen speciális operátor osztályokra vonatkozó jellemzési, faktorizációs, illetve kiterjesztési tétel bemutatása. A disszertáció eredményei a szerző [52, 53, 60, 61, 62, 63] publikációiból, illetve az [54, 55] kéziratokból valók.

Míg a mindenütt definiált folytonos lineáris operátorok között a szimmetrikus és az önadjungált operátor fogalma egybe esik, nemkorlátos operátorok körében az önadjungált-ság meghatározóbb tulajdonság a szimmetrikusságnál. A két fogalom közti különbséget jól szemlélteti, hogy sűrűn definiált szimmetrikus operátor négyzete lehet triviális, azaz csupán a zéró altérben értelmezett. Önadjungált operátor négyzete viszont Neumann egy klasszikus tételének értelmében nem csak hogy sűrűn definiált, de maga is önadjungált. A nemkorlátos operátorok elméletének egyik leghatékonyabb eszköze, a spektrál tétel is azt mutatja, hogy valójában operátor kalkulust csak normális, speciálisan önadjungált operátorok körében lehet értelmezni.

Az önadjungált operátorok gazdag irodalmából itt csupán Neumann János [36] cikkét emeljük ki, amelyben Neumann a sűrűn definiált, zárt szimmetrikus operátorok úgynevezett defekt indexét vizsgálva elsőként adta meg az önadjungált-ság spektrális jellemzését, illetve az önadjungált kiterjeszthetőség feltételét. Az első két fejezetben Neumann eredményeinek általánosításait adjuk meg. Az általunk prezentált eredmények hatékonysága abban rejlik, hogy a szóban forgó operátor önadjungált-ságának igazolásához nincs szükség az értelmezési tartomány sűrűségét feltenni; ez automatikus következménye feltevéseinknek. Másfelől, a Neumann által adott eredmények kizárólag komplex Hilbert térben érvényesek. Az általunk bizonyított karakterizációs tételek viszont lehetővé teszik a valós és komplex eset együttes tárgyalását. Alkalmazásként igazoljuk a Kato–Rellich, illetve a Wüst perturbációs tételeket valós és komplex Hilbert tér fölött.

Szintűgy Neumann nevéhez fűződik a következő nevezetes tétel: ha T Hilbert terek között ható sűrűn értelmezett zárt lineáris operátor, akkor a T^*T operátor önadjungált. Ugyanakkor egyszerű példa illusztrálja, hogy itt a T operátor zártaságára vonatkozó feltétel nem szükséges, de nem is hagyható csak úgy el. A 3. fejezetben igazoljuk, hogy T^*T -nek mindig létezik önadjungált kiterjesztése, illetve szükséges és elégséges feltételt adunk a T^*T alakú operátorok önadjungált-ságára. Végül, Friedrichs [18] egy klasszikus

eredményét általánosítva megmutatjuk, hogy tetszőleges A sűrűn definiált pozitív operátor előáll T^*T faktorizációs alakban valamely T lezárható operátorra, továbbá ekkor T^*T^{**} megegyezik A Friedrichs kiterjesztésével.

A 4. fejezetben szuboperátorok egy speciális alosztályát vizsgáljuk: jellemezzük azokat az operátorokat, amelyeknek létezik zárt képterű pozitív önadjungált kiterjesztése. Kiderül, hogy ez a feltétel ekvivalens azzal, hogy az illető operátor legkisebb, ún. Krein–von Neumann kiterjesztése zárt képterű. Végül Sebestyén és Stochel [49] faktorizációs módszerének segítségével jellemezzük a zárt képterű extrémális kiterjesztéseket, illetve megadjuk a Krein–von Neumann kiterjesztés Moore–Penrose-féle pszeudoinverzét.

Az 5. fejezetben Matolcsi és Farkas [16] munkáját folytatva rövid és egyszerű faktorizációs konstrukciót adunk pozitív operátorok formaösszegére. Eljárásunk segítségével jellemezzük a formaösszeg operátornak, illetve négyzetgyökének értelmezési tartományát, mag- illetve képterét. Végül, az előző fejezet eredményeit felhasználva szükséges és elégséges feltételt adunk a formaösszeg képterének zártságára, továbbá faktorizációs előállítását adjuk a formaösszeg Moore–Penrose-féle pszeudoinverzének.

A 6. fejezetben szimmetrizálható operátorok biortogonális sorfejtettségével foglalkozunk. A fejezet fő eredményeként Krein [34] egy ide vonatkozó alaptételét általánosítjuk.

A 7. fejezetben új megvilágításba helyezzük Ando [2] pozitív operátorok Lebesgue-felbontásáról szóló tételét az operátor faktorizáció, illetve a lineáris relációk elméletének segítségével. Megmutatjuk továbbá, hogy egy pozitív operátor egy másik pozitív operátorra vett abszolút folytonossága, szingularitása, illetve a Lebesgue-felbontás egyértelműsége szoros kapcsolatban van egy bizonyos lineáris reláció lezárhatóságával, szingularitásával, illetve folytonosságával.